WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 6: The Haar Filter Bank

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1 Introduction

In this lecture our aim is to implement Haar MRA using appropriate filter banks. In the analysis part we decompose given function in y_{v0} and y_{w0}. Decomposition of corresponding sequence is carried out in terms of wavelet function $\psi(t)$ and scaling function $\phi(t)$. After this is done we explore signal reconstruction using y_{v0} and y_{w0}. Earlier, we have seen that we can divide the real axis (number line) into various equally spaced blocks which then constitute a space. For example, let us split the line into blocks of width 1 as shown below. Corresponding

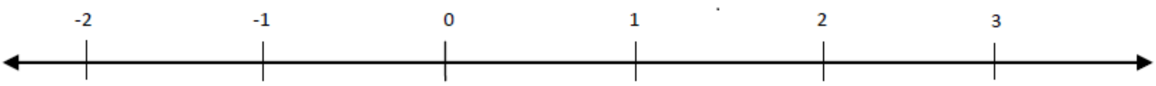

Figure 1: Number Line

to Figure 1, we have basis functions of V_0 and W_0. If the same functions were used over a period of half ($\frac{1}{2}$), we have the spaces V_1 and W_1 and by halving or doubling the width we achieve the whole range of spaces.

$$\{0\} \subset \ldots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots$$

2 Analysis part

Once we have decided the space in which we are operating, we can create a piecewise approximation of given function. Consider a function $y(t) \in V_1$ defined between [-1, 3]. The number written in between the interval indicates piecewise constant amplitude for that interval.

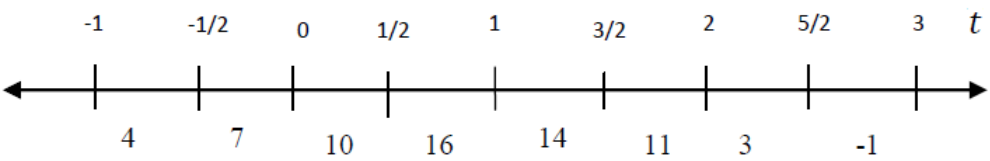

Figure 2: $y(t) \in V_1$

This continuous function can be associated with the sequence

$$y[-2] = 4,\ y[-1] = 7,\ y[0] = 10,\ y[1] = 16,\ y[2] = 14,\ y[3] = 11,\ y[4] = 3,\ y[5] = -1$$

$y[n]$ is an approximation of $y(t)$ over period $[\frac{n}{2}, \frac{n+1}{2}]$ in the space V_1.
In general, we write

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]\phi(2t-n)$$

where $\phi(2t-n)$ is shifted version of basis function of V_1 shown in figure 3.

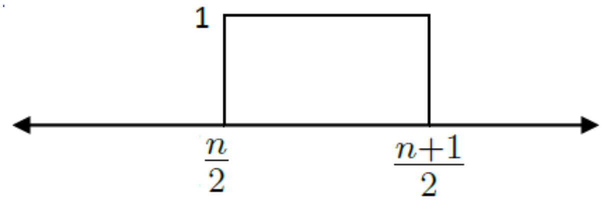

Figure 3: Basis function of V_1

Now we decompose space V_1 in two subspaces V_0 and W_0 as

$$V_1 = V_0 \bigoplus W_0$$

Corresponding to these subspaces we obtain two functions y_{v0} and y_{w0}. So, for a given $y(t) \in V_1$ we can split it in two components $y_{v0}(t)$ and $y_{w0}(t)$ which are projections of $y(t)$ on the V_0 and W_0 subspaces, respectively.
Note that this is after the whole analysis part (after the decimation operation). These functions can be represented as shown in figure 4.

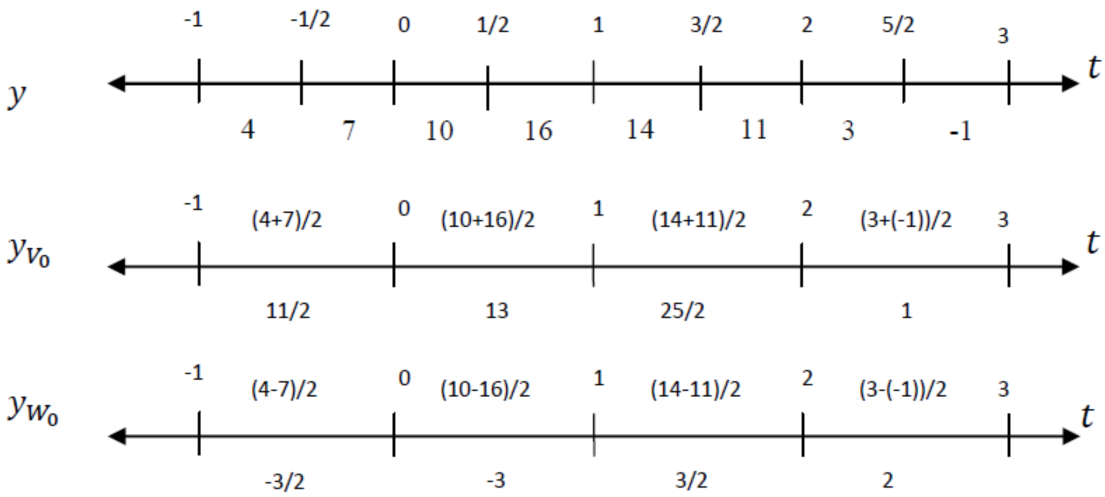

Figure 4: Projection of $y(t)$ over subspaces V_0, W_0

The above representations graphically demonstrate scaled and shifted combinations of the bases to get the original signal (sequence).
We define $a[n]$ as the input and $b_1[n]$ and $b_2[n]$ as the output.

$$b_1[n] = \frac{1}{2}(a[n] + a[n-1])$$

$$b_2[n] = \frac{1}{2}(a[n] - a[n-1])$$

This is equivalent to

$$b_1[n] = \frac{1}{2}(y[n] + y[n-1])$$

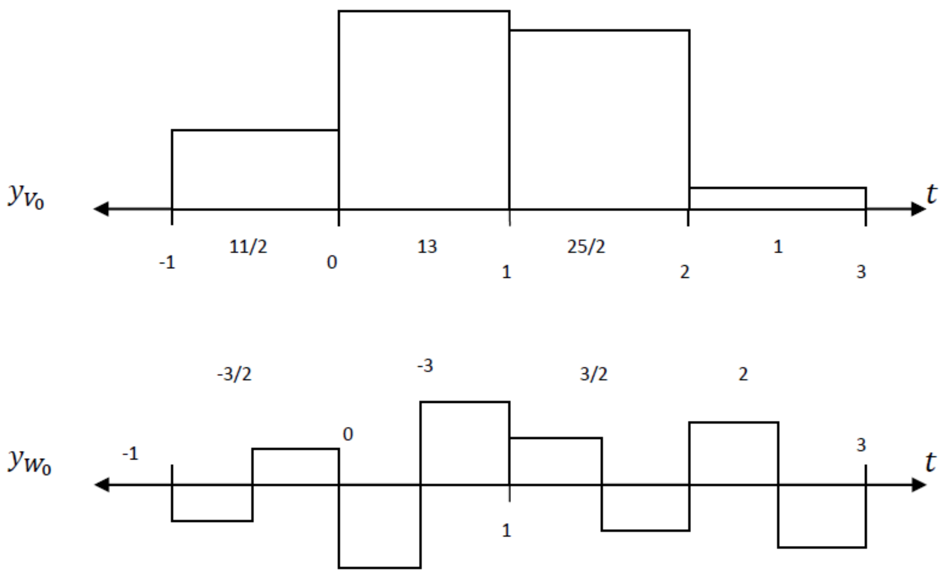

Figure 5: Graphical representation of y_{v0} and y_{w0}

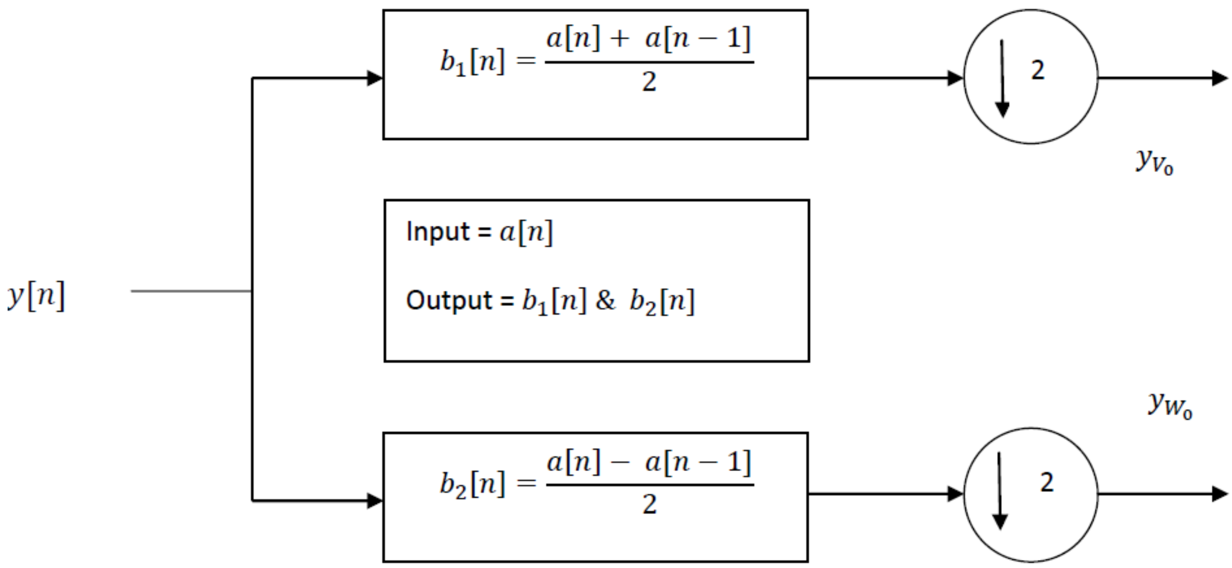

Figure 6: Filter Bank: Analysis Part

$$b_2[n] = \frac{1}{2}(y[n] - y[n-1])$$

Here, $b_1[n]$ and $b_2[n]$ are sequences having the same length and order as $y[n]$ whereas we want them to be shorted and one order lesser than $y[n]$. $b_1[n]$ and $b_2[n]$ must be modified somehow to get $y_{v0} \in V_0$ and $y_{w0} \in W_0$ respectively. This is performed by decimation (down-sampling). Taking Z-transform on both the sides, we get

$$B_1(Z) = \frac{1}{2}(1 + z^{-1})Y(Z)$$

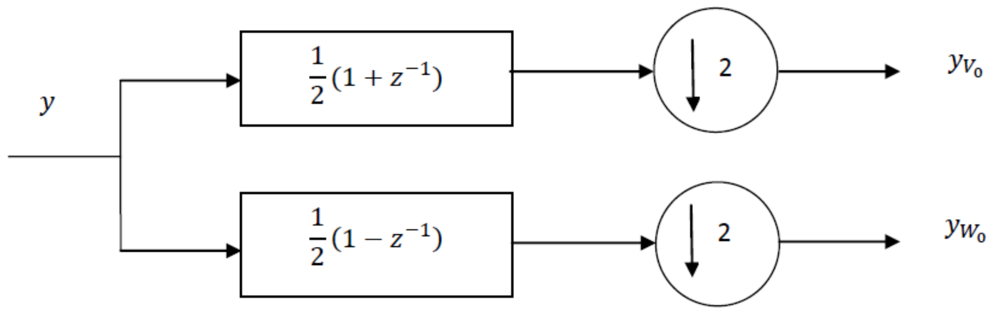

Figure 7: Filter Bank: Analysis Part

$$B_2(Z) = \frac{1}{2}(1 - z^{-1})Y(Z)$$

This is followed by the decimation operation to remove unwanted data. Hence, the Analysis filter bank is as shown in figure 6 .

3 Synthesis Part

To synthesize $y(t)$ from $y_{v0}(t)$ and $y_{w0}(t)$, in continuous time, we can write $y(t) = y_{v0}(t)+y_{w0}(t)$, however it is required to do some work in discrete time. We again write all the three sequences as,

$$y[n] = \{4, 7, \underset{\uparrow}{10}, 16, 14, 11, 3, -1\}$$

$$y_{v0}[n] = \{\frac{11}{2}, \underset{\uparrow}{13}, \frac{25}{2}, 1\}$$

$$y_{w0}[n] = \{\frac{-3}{2}, \underset{\uparrow}{-3}, \frac{3}{2}, 2\}$$

Upsampler: To 'outdo' or 'overcome' decimation operation, we define operation of upsampling by symbol

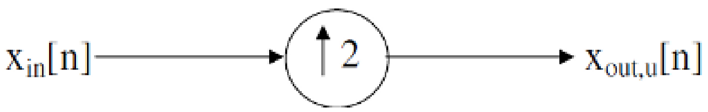

Figure 8: Upsampler

$$\begin{aligned} x_{out}[n] &= x_{in}\left[\frac{n}{2}\right], \qquad where\, n\, is\, multiple\, of\, 2 \\ &= 0, \qquad otherwise \end{aligned}$$

Basically upsampler expands the input sequence and it does so by adding zero in between successive samples.

If $x_{in}[n] = y_{v0}[n]$, then x_{out} is given by

$$x_{out}[n] = \{\frac{11}{2}, 0, \underset{\uparrow}{13}, 0, \frac{25}{2}, 0, 1, 0\}$$

and similarly, $y_{w0}[n]$ on upsampling gives,

$$x_{out}[n] = \{\frac{-3}{2}, 0, \underset{\uparrow}{-3}, 0, \frac{3}{2}, 0, 2, 0\}$$

If the sequences obtained after upsampling (y_{v0} and y_{w0}) are added and subtracted alternately, we can recover original sequence. We can combine the operations with proper up-sampling and delay to get back the original . Figure 9 shows how an operation of upsampling is done using signal flow diagram. The delay is used to give the output in proper order.

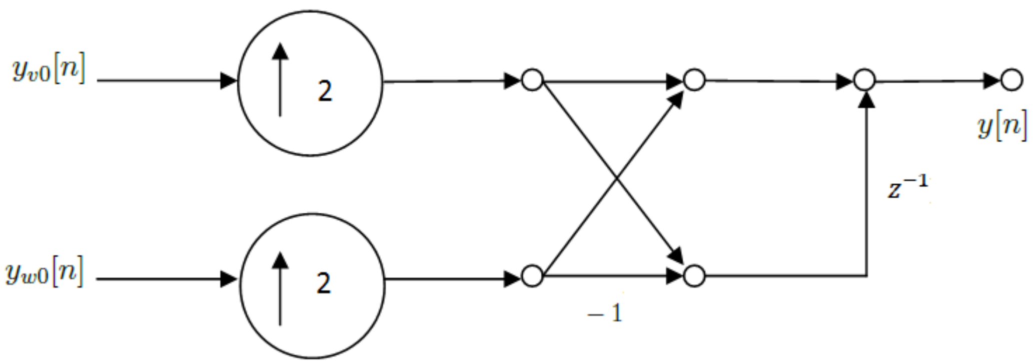

Figure 9: Signal flow diagram with up-sampler by two

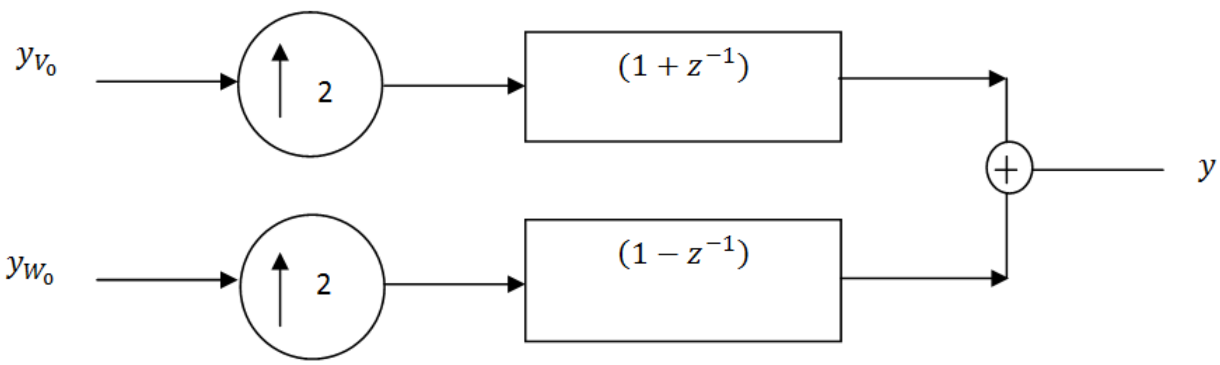

Figure 10: Filter Bank: Synthesis Part

Figure 10 shows the synthesis part of 2-band perfect reconstruction filter bank.
In this way, filter bank is used to implement Analysis and Synthesis aspects of Haar MRA.