WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 5: Equivalance between Function and Sequence

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1 Introduction

In the previous lecture, we saw an equivalence between functions and vectors w.r.t. inner product and Parsevals theorem. We saw how functions can be considered as generalized vectors. Another dimension of same is replacing work with functions with work with sequences. It is possible to work with sequences in place of functions. Sequences are much easier to deal with the computer. It can be processed in discrete time by a computer that further produces a sequence. If whatever we are doing with a sequence maps exactly with what we want do with an original continuous time functions then it is an added advantage. This is true for the spaces contained in V_0 contained in V_1 contained in V_2 and so on in the $\mathbb{L}_2(\mathbb{R})$ ladder shown below.

$$\{0\} \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \dots$$

Basis functions for V_0 subspace is given by $\{\Phi(t-n)\}_{n\in\mathbb{Z}}$. The function $\Phi(t-n)$ for $t = n$ is shown below. This is also an orthonormal basis for V_0 as

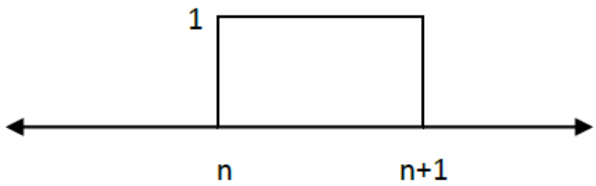

Figure 1: Basis function for V_0

$$\begin{aligned}\langle \Phi(t-n), \Phi(t-m) \rangle &= 0, \qquad n \neq m \\ &= 1, \qquad n = m\end{aligned}$$

where $n, m \in \mathbb{Z}$.

2 Function and Sequence

To have an idea between function and sequence consider the function $x(t) \in V_0$

$$x(t) = \dots + \left(\frac{1}{2}\right)\Phi(t+1) + \left(\frac{-3}{4}\right)\Phi(t) + \left(\frac{3}{2}\right)\Phi(t-1) + (4)\Phi(t-2) + \dots$$

There is an equivalence between $x(t)$ and sequence. So, corresponding sequence is

$$x[n] = [\ldots, \frac{1}{2}, -\underset{\uparrow}{\frac{3}{4}}, \frac{3}{2}, 4, \ldots]$$

The function $x(t)$ belongs to $V_0 \in \mathbb{L}_2(\mathbb{R})$ and thus sequence also belongs to a set of square summable sequences. If a function $x(t)$ is square integrable i.e.$(\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty)$ then corresponding sequence is square summable i.e. $(\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty)$. Here, we use a notion that $x(t) \in \mathbb{L}_2(\mathbb{R})$, implies $x[n] \in l_2(\mathbb{Z})$.
In general, $l_p(\mathbb{Z})$ is a linear space of sequences such that

$$(\sum_{n=-\infty}^{\infty} |x[n]|^p < \infty)$$

We have just shown a correspondence that if $x(t) \in V_0 \in L_2(\mathbb{R})$ then $x[n] \in l_2(\mathbb{Z})$. Note x[n] is the sequence of coefficients of expansion of x(t) with respect to an ORTHONORMAL basis. If the basis is orthonormal then there is mapping between inner products. For example, suppose $x(t), y(t) \in V_0$,then inner product in continuous time is given by

$$\begin{aligned} \langle x(t), y(t) \rangle &= \int_{-\infty}^{\infty} x(t)\overline{y(t)}dt \\ &= K_0 \sum_{n=-\infty}^{\infty} x[n]\overline{y[n]} \end{aligned} \tag{1}$$

where K_0 is a constant. Therefore, what we do in context of continuous time function can be equivalently done in the context of discrete domain for corresponding sequence. So, eventually we are forgetting continuous functions and dealing with sequence $x[n]$. *What is the motivation behind this?* Our motive is to extract an incremental information from going one subspace to another in a ladder. Now we will see how to move from one resolution to another or extract the incremental information from the function. Note that this process corresponds to going from low resolution subspaces to high resolution in $L_2(\mathbb{R})$ ladder of subspaces. Consider $y(t) \in V_1$. Corresponding sequence y[n] is

$$[\ldots, 4, 7, 10, \underset{\uparrow}{16}, 11, 3, -1 \ldots]$$

The relationship between y(t) and y[n] is

$$y(t) = \sum_{n \in \mathbb{Z}} y[n]\Phi(2t - n)$$

where

$$\begin{aligned} \Phi(2t - n) &= 1, \qquad \frac{n}{2} < t < \frac{n+1}{2} \\ &= 0, \qquad \textit{otherwise} \end{aligned}$$

Now consider an orthogonal decomposition of a function in V_1, into functions in V_0 and W_0

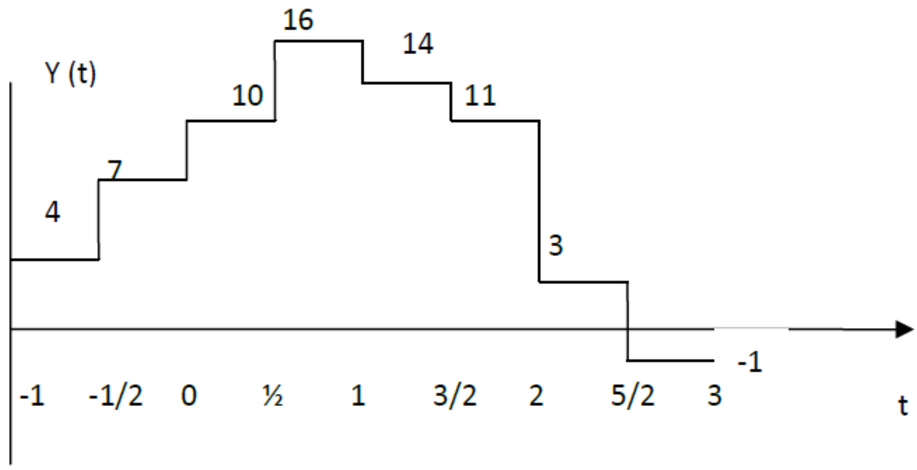

Figure 2: y(t)

denoted by

$$V_1 = V_0 \bigoplus W_0$$

where

$$W_0 = span\{\Psi(t-n)\}_{n\in\mathbb{Z}}$$

$$V_0 = span\{\Phi(t-n)\}_{n\in\mathbb{Z}}$$

V_1 is the orthogonal sum of subspaces V_0 and W_0. The idea of orthogonal decomposition is to decompose a function into functions of smaller spaces such that decomposed components are orthogonal to each other or their inner product is zero. For example, a room can be considered as an orthogonal sum of 2-d floor and a multiple of 1-d vectors perpendicular to the floor. Any vector in the plane of the floor is perpendicular to the vectors which are perpendicular to the floor and their inner product will also be zero. The same concept can be generalized to N-dimensional space.

We have seen the equivalence between functions and sequences with respect to square integrability and inner products. Now consider the angle between two functions or vectors defined in terms of their inner product. There exists equivalence with respect to angles also. As we are considering functions and sequences as generalized vectors, angle between two functions $x(t)$ and $y(t)$ in any subspace is defined by

$$\cos(\theta) = \frac{\langle x(t), y(t)\rangle}{\parallel x \parallel \parallel y \parallel}$$

where θ is angle between $x(t)$ and $y(t)$ and $x(t), y(t) \in L_2(\mathbb{R})$.

To check whether it is possible to decompose a sequence in terms of other sequences, consider $x_1(t) \in V_0$ and $x_2(t) \in W_0$ denoted by solid and dashed lines respectively, as shown in figure 3 below. Note that $\langle x_1(t), x_2(t)\rangle = 0$ and thus $x_1(t)$ and $x_2(t)$ are orthogonal to each other.

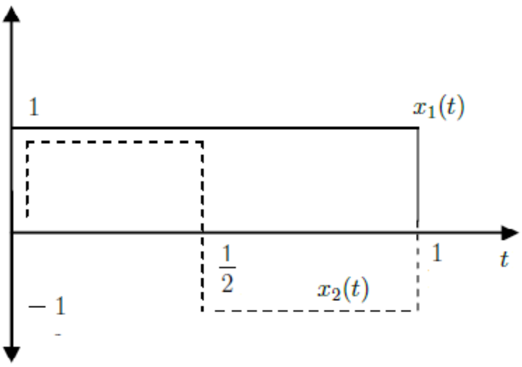

Figure 3: Functions $x_1(t)$ and $x_2(t)$

Take any function in V_1 in the open interval $]n, n+1[$. It can be written as a summation of function belonging to V_0 and and a function belonging to W_0 from figure 3. Thus we can decompose a function in V_1 in functions in V_0 and W_0 in a unique way. *Can we make corresponding construction on the sequences?*
Consider

$$V_1 \colon p(t) \longrightarrow p[n]$$
$$W_0 \colon q_0(t) \longrightarrow q_0[n]$$
$$V_0 \colon p_0(t) \longrightarrow p_0[n]$$

here $p_0(t)$ and $q_0(t)$ are orthogonally decomposed functions of $p(t)$ thus, $p(t) = p_0(t) + q_0(t)$ and p[n], $p_0[n]$ and $q_0[n]$ are corresponding sequences. To check whether $p[n] = p_0[n] + q_0[n]$ holds or not, consider the three sequences p[n], $p_0[n]$ and $q_0[n]$ of V_1, V_0 and W_0 subspaces. If unit interval in V_0 and W_0 subspaces is $]n, n+1[$ then the corresponding interval in V_1 and W_1 subspaces is $]2n, 2n+2[$ and the previous example is reconsidered for the corresponding sequences in the figures below. Note that function in V_1 has the value C1 at 2n and C2 at

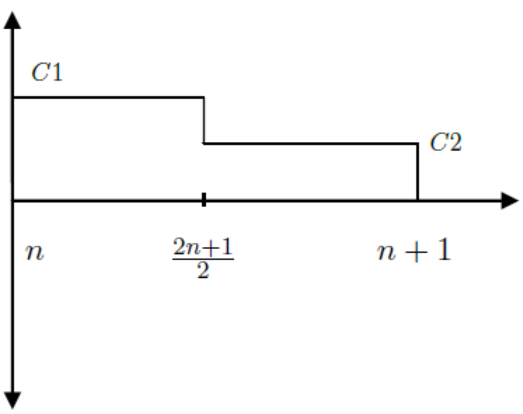

Figure 4: Sequence corresponding to function $p(t) \in V_1$

(2n+1). Thus, $p[2n] = C1$, $p[2n+1] = C2$.
Similarly,

$$p_0[n] = \frac{(C1 + C2)}{2}$$

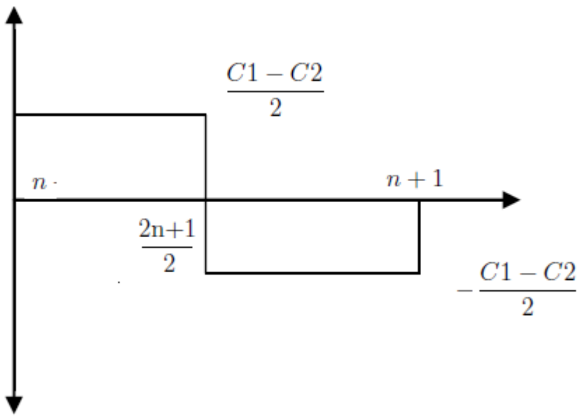

Figure 5: Sequence corresponding to function $q_0(t) \in W_0$

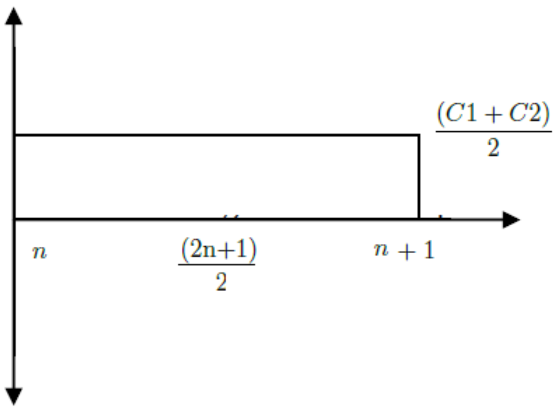

Figure 6: Sequence corresponding to function $p_0(t) \in V_0$

$$q_0[n] = \frac{C1 - C2}{2}$$

Note that the relation among p[n], $p_0[n]$ and $q_0[n]$ is not $p[n] = p_0[n] + q_0[n]$, but

$$p_0[n] = \frac{p[2n] + p[2n+1]}{2} \tag{2}$$

$$q_[n] = \frac{p[2n] - p[2n+1]}{2} \tag{3}$$

Above equations show that $p_0[n]$ and $q_0[n]$ are outputs of discrete time filters. Then the Discrete time filter with x[n] and y[n] as input and output respectively is given by

$$y[n] = \frac{x[n] + x[n+1]}{2} \tag{4}$$

3 Downsampler

In the previous example if p[n] is input to the filter given by equation (4), then output is not $p_0[n]$. For output to be equal to $p_0[n]$, the filter must be driven by a system with input x_{in} and

output x_{out} related by $x_{out}[n] = x_{in}[2n]$.
Now consider the system which does this operation. It retains only the even samples of the input sequence and removes the odd samples. Note that it locates the even samples at half of the original sample number. For example if $x_{in}[n] = [6\ 3\ 5\ 2\ 7\ 8\ 3\ 4]$ is the input to this system then output $x_{out}[n] = [6\,5\,8\,4\,]$. Note that $x_{out}[0]$ comes from $x_{in}[0]$, $x_{out}[2] = x_{in}[4]$ and thus $x_{out}[n] = x_{in}[2n]$. Such a system is called **Downsampler** or **Decimator**. Figure 7 shows decimation by 2.

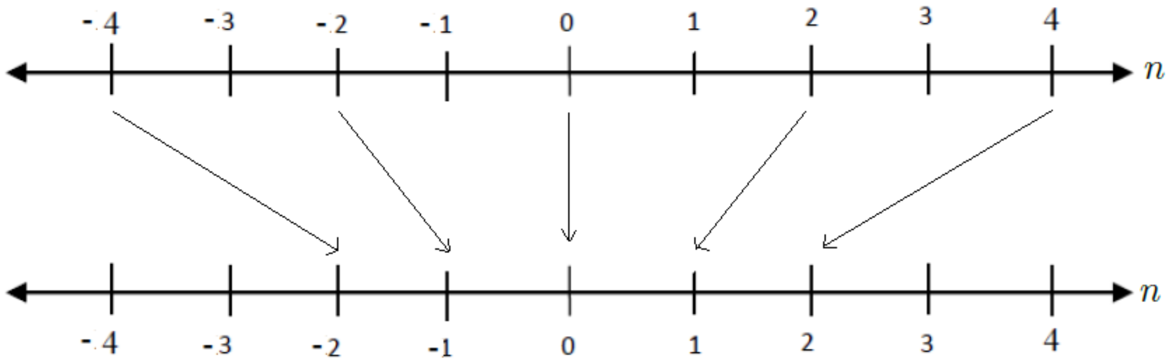

Figure 7: Downsampling by 2

Thus to implement the equation (2) we need a discrete time filter given by equation (4) followed by a downsampler by 2. This helps to construct a sequence $p_0[n]$ from $p[n]$.