WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 4: Algebra of Linear Vector Spaces, Bases, etc

Prof. V.M.Gadre, EE, IIT Bombay

1 Generalized vectors:

A vector quantity or vector, provides the magnitude as well as the direction of a specific quantity.

Example: When giving directions to a point, it is not enough to say that it is x miles away, but the direction of those x miles must also be provided for the information to be useful. (Note that physical quantities are represented by Scalars, such as temperature, volume and time etc.)

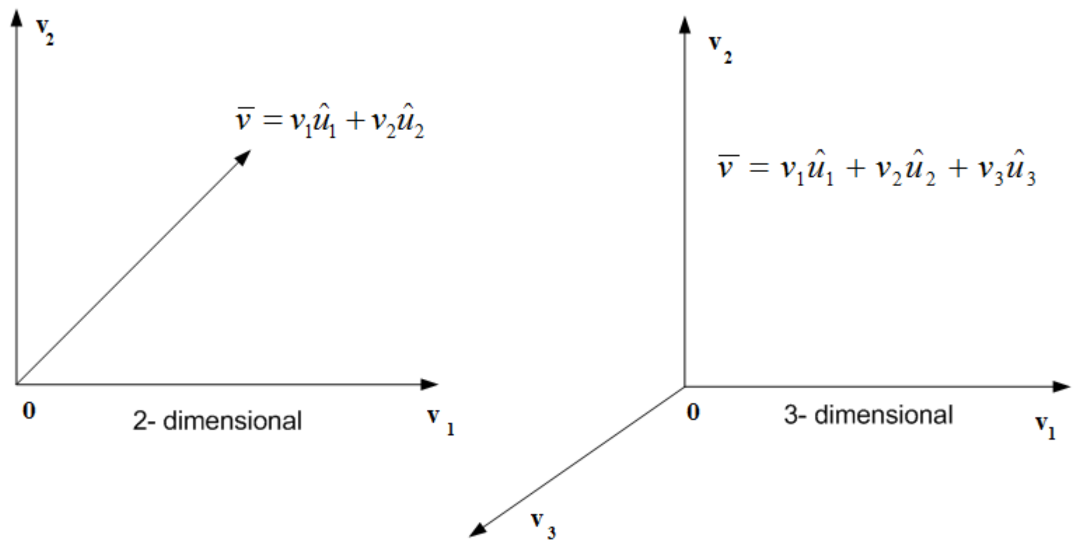

Figure 1: Graphical representation of vectors

Given a coordinate system in three dimensions, a vector may thus be represented by an ordered set of three components which represent its projections v_1, v_2, v_3 on the three coordinate axes.

$$v = [v_1, v_2, v_3]$$

The three most commonly used coordinate systems are rectangular, cylindrical, and spherical. Alternatively, a vector may be represented by the sum of the magnitudes of its projections on three mutually perpendicular axes:

$$\bar{v} = v_1\hat{u}_1 + v_2\hat{u}_2 + v_3\hat{u}_3$$

The n-dimensional coordinate systems based on the Euclidean space (Cartesian space or n-space) represented by R^n or E^n, under n-dimensions and n-vectors. Usually, the Euclidean space is formed by $(X_1, X_2, X_3, ..., X_n)$ where n is equal to 8.

Parallelogram law of vector:
Let us take an example, in this using parallelogram law we can get the resultant vector. The resultant vector can be calculated as:

$$\begin{aligned} \bar{v} &= \bar{\bar{v}}_1 + \bar{\bar{v}}_2 \\ \text{where } \bar{\bar{v}}_1 &= k_1\hat{u}_1 \\ \text{and } \bar{\bar{v}}_2 &= k_2\hat{u}_2 \\ \text{then } \bar{v} &= k_1\hat{u}_1 + k_2\hat{u}_2 \end{aligned}$$

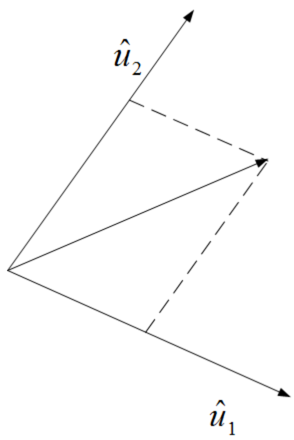

Figure 2: Parallelogram law of vectors

2 Relationship between functions, sequences, vectors:

One can intimately relate processing of a function to processing of equivalent sequence, and whatever we are doing to try and gain information from or modify a function can be done equivalently by processing or modifying that sequence corrosponding to function. A sequence is like a vector and each n is a different dimension of that vector.
An infinite (countably infinite) dimension vector is a sequence $x[n], n \in \mathbb{Z}$, where n is index and $\mathbb{Z}$ is set of integers.
Now, we would like to extend other ideas of vectors to this context of infinite dimension vector.

Dot product of vectors:
Let,
$\bar{e_1} = e_{11}\hat{u}_1 + e_{12}\hat{u}_2$ and $\bar{e_2} = e_{21}\hat{u}_1 + e_{22}\hat{u}_2$ then dot product is $\bar{e_1}\bar{e_2} = e_{11}e_{21} + e_{12}e_{22}$. That means it is nothing but sum of products of corresponding coordinates.

Let two n-dimensional vectors as
$\bar{e_1} : e_{11}, e_{12}, ..., e_{1N}$ and $\bar{e_2} : e_{21}, e_{22}, ..., e_{2N}$ the dot product of these two vectors is $\langle \bar{e_1}\bar{e_2} \rangle = \sum_{k=1}^{N} e_{1k}e_{2k}$. These are also called as orthogonal coordinates.

Let two sequences, say $x_1[n], x_2[n], n \in Z$, the 'dot product' or 'inner product' is $\langle x_1, x_2 \rangle$, where

$$\langle x_1, x_2 \rangle = \sum_{n=-\infty}^{+\infty} x_1[n] x_2[n]$$

In $2-D, 3-D$ space, we will calculate magnitude from the dot product, but in general $n-D$ space, we will use norm. Generally norm squared represents energy.
Let vector x: essentially a sequence $x[n], n \in Z$, then the 'norm' of sequence $x = \|x\|$ should be $\langle x_1, x_2 \rangle^{1/2}$

$\|x\| \geq 0$ and$\|x\| = 0$ iff $x = 0$ i.e. $x[n] = 0 \ \forall \ n \in Z$

If x_1 and x_2 are real,

$$\begin{aligned} \langle x_1, x_2 \rangle &= \sum_{n=-\infty}^{+\infty} x_1[n] x_2[n] \\ \langle x, x \rangle &= \sum_{n=-\infty}^{+\infty} x^2[n] \end{aligned}$$

As long as $x[n]$ is real $\forall n \in \mathbb{Z}$, this will satisfy norm requirements.
A small change will be applied for complex sequences as follows

$$\langle x_1, x_2 \rangle = \sum_{n=-\infty}^{+\infty} x_1[n] \overline{x_2[n]}$$

Properties of Inner product:

1.Conjugate community:

$$\begin{aligned} \langle x_1, x_2 \rangle &= \overline{\langle x_2, x_1 \rangle} \\ &= \sum_{n=-\infty}^{+\infty} x_1[n] \overline{x_2[n]} \\ &= \sum_{n=-\infty}^{+\infty} x_2[n] \overline{x_1[n]} \\ \langle x_1, x_2 \rangle &= \overline{\langle x_2, x_1 \rangle} \end{aligned}$$

2. Linear in first argument :

$$\begin{aligned} \langle a_1 x_1 + a_2 x_2, x_3 \rangle &= a_1 \langle x_1, x_3 \rangle + a_2 \langle x_2, x_3 \rangle \\ &= \sum_{n=-\infty}^{+\infty} (a_1 x_1 + a_2 x_2) x_3 \\ &= \sum_{n=-\infty}^{+\infty} a_1 (x_1 x_3) + a_2 (x_2 x_3) \\ \langle a_1 x_1 + a_2 x_2, x_3 \rangle &= a_1 \langle x_1, x_3 \rangle + a_2 \langle x_2, x_3 \rangle \end{aligned}$$

3. Positive definite :

$$\begin{aligned}\langle x, x\rangle &= \sum x[n].x[n] \\ \langle x, x\rangle &= 0; \quad iff \quad x[n] = 0 \quad \forall n\end{aligned}$$

Extension to uncountably infinite dimension:
For any 't', $t \in \mathbb{R}$ is a different dimension and $x(t)$, $t \in \mathbb{R}$, means $x(t)$ for a particular 't^{th}' coordinate. Then the 'dot product' or 'inner product' between two functions $x(t)$ and $y(t)$ is

$$\langle x, y\rangle = \int_{-\infty}^{+\infty} x(t)y(t)dt$$

Parseval's Theorem:
The Parseval's theorem states that the inner product of any two functions in time domain is equal to the inner product of those two functions in frequency domain.
Let $x(t)$ be a function and $\widehat{x}(\nu)$ or $\widehat{x}(\Omega)$ is its fourier transform (in Hz or in radians) and defined as

$$\widehat{x}(\nu) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi\nu t}dt \quad or \quad \widehat{x}(\Omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\Omega t}dt \quad where \quad \Omega = 2\pi\nu$$

Let $y(t)$ be a function and $\widehat{y}(\nu)$ or $\widehat{y}(\Omega)$ is its fourier transform (in Hz or in radians) and defined as

$$\widehat{y}(\nu) = \int_{-\infty}^{+\infty} y(t)e^{-j2\pi\nu t}dt \quad or \quad \widehat{y}(\Omega) = \int_{-\infty}^{+\infty} y(t)e^{-j\Omega t}dt \quad where \quad \Omega = 2\pi\nu$$

The inner product of these in time domain is

$$\langle x, y\rangle = \int_{-\infty}^{+\infty} x(t)\overline{y(t)}dt$$

and it is equal to the inner product in frequency domain given by

$$\langle \widehat{x}, \widehat{y}\rangle = \int_{-\infty}^{+\infty} \widehat{x}(\nu)\overline{\widehat{y}(\Omega)}d\Omega$$

That means $\langle x, y\rangle = \langle \widehat{x}, \widehat{y}\rangle$
The function $x(t)$ can be reconstructed from its frequency components as

$$x(t) = \int_{-\infty}^{+\infty} \widehat{x}(\Omega)e^{-j\Omega t}d\Omega$$

Applications of Parseval's Theorem:
The Parseval's theorem is often used in many areas like physics and engineering etc, and it is written many of the times as

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |\widehat{x}(\nu)|^2 d\nu$$

where $\widehat{x}(\nu)$ represents the continuous Fourier transform of $x(t)$ and 'ν' represents the frequency component of x.

From this equation, the theorem tells that the total energy contained in a function $x(t)$ over all time 't' is equal to the total energy of the its Fourier Transform $\widehat{x}(\nu)$ over all frequency 'ν'.

For discrete time signals, the theorem becomes:

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} |\widehat{x}(e^{j\omega})|^2 d\omega$$

where $\widehat{x}(e^{j\omega})$ is the Discrete-Time Fourier transform (DTFT) of x and 'ω' represents the angular frequency (in radians per sample) of x.

For the Discrete Fourier transform (DFT), the relation becomes:

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |\widehat{x}[k]|^2$$

where $\widehat{x}[k]$ is the DFT of $x[n]$ and 'N' is length of sequence in both domain .

Relation between continuous functions and sequences:
Let $x(t)$ be a continuous function and let $\phi(t)$ be a unit step function in $[0, 1]$ interval, then $x(t)$ can be written as

$$x(t) = ... + C_{-1}\phi(t+1) + C_0\phi(t) + C_1\phi(t-1) + C_2\phi(t-2) + ...$$

It can be graphically represented as shown in figure 3.

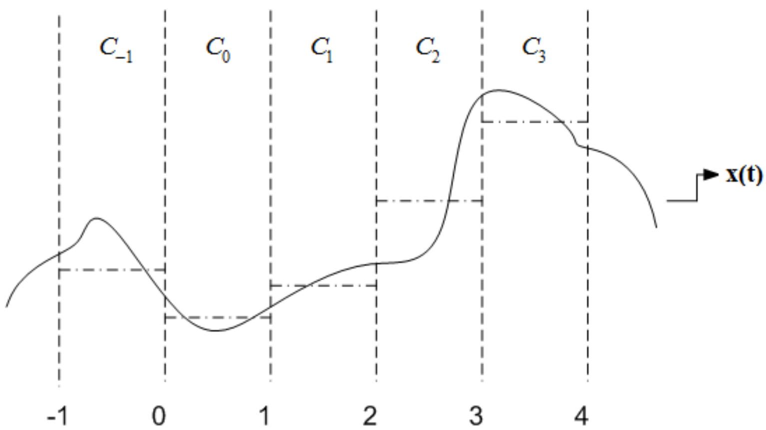

Figure 3: Relation between continuous functions and sequences

Equivalance between continuous functions and sequences will be dealt in greater detail in subsequent lectures.