WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 3: Haar MRA (Multi Resolution Analysis)

Prof.V.M.Gadre, EE, IIT Bombay

1 Introduction

The underlying principle of wavelets is to capture incremental information in a function. The piecewise constant approximation of a function is representation of the function at different resolutions.

For understanding this, consider an example of a cabbage. Let the outermost shell be the maximum resolution. The job of wavelet is to *peel-off* or to take out a particular shell of that cabbage. So we are essentially peeling-off shell-by-shell using different dilates and translates of a wavelet. Figure 1 shows the concept of nested subspaces. So, it goes like this,

$$\text{Different dilates} \xrightarrow{\text{corresponds to}} \text{different resolutions.}$$
$$\text{Different translates} \xrightarrow{\text{takes us along}} \text{different resolutions.}$$

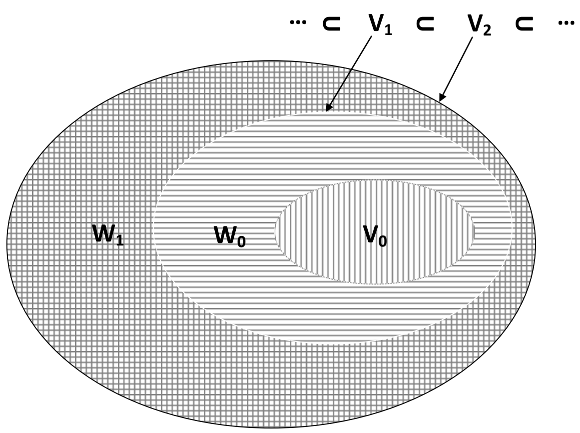

Figure 1: Nested Subspaces

The idea of wavelets may be introduced using an example of the Haar wavelet. The Haar wavelet is a dyadic wavelet, that is, the piecewise constant approximation is refined in steps of two at a time. The wavelet captures the incremental information between two consecutive levels of resolution. In other words, the Haar wavelet gives the additional information required to go from one resolution to the next higher level of resolution.

Example 1

The idea of expressing a function at different resolutions may be explained with the example of Figure 2. It shows the signal at high resolution in red, its piecewise constant approximation over unit intervals in blue and that over intervals of length 0.5 in green.
The corresponding function which gives the incremental information between the two approximation levels is shown in Figure 3. The same idea may be extended to two dimensions as well.

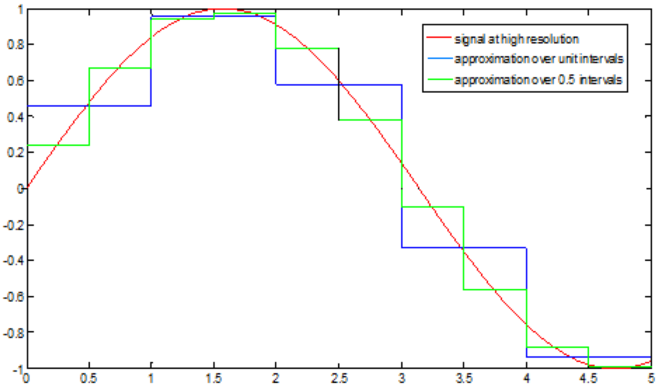

Figure 2: Signal at different Resolution

In Figure 4, Figure 4a is the image at a certain level of resolution. Figure 4b(i) is the image at 0.5 resolution of Figure 4a. Figure 4b(ii), 4b(iii) and 4b(iv) give the incremental information in the vertical, horizontal and diagonal directions respectively.

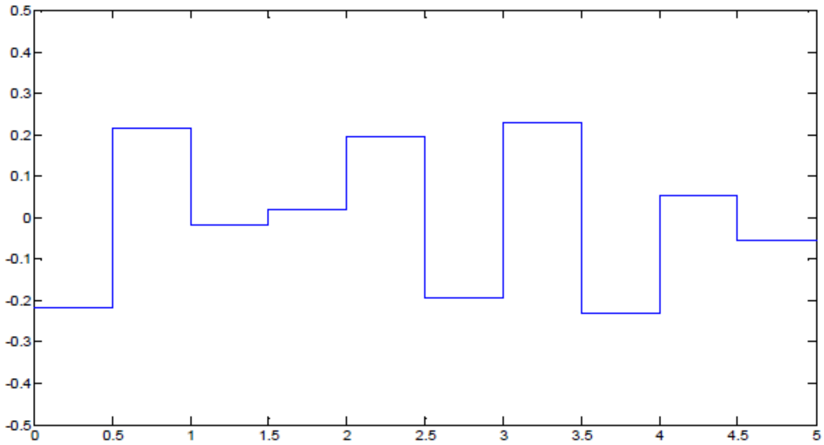

Figure 3: Incremental information

The idea of wavelets is analogous to an object with many shells. Wavelet translates at the maximum resolution takes out the outermost shell, the next shell is taken out at the next lower resolution and so on. Hence, we are essentially 'peeling off' shell by shell using different dilates and translates of the wavelet function. The dilation takes us to the next level of resolution, while translation takes us along a given resolution.
Without any loss of generality, we begin with piecewise constant approximation at a resolution of unit length. The choice of unit interval is entirely one's own choice. We now need to find a function $\phi(t)$ such that its integer translates are able to span the space of piecewise constant functions on the standard unit intervals. Here, space refers to a linear space of functions, that is, a set of functions which is closed under linear combinations. In this discussion we only consider finite linear combinations. The same ideas may be extended for infinite linear combinations.
A set V_0 is defined as follows,
$V_0 : \{ \ x(t)$, such that $x(\cdot) \in L_2(\mathbb{R})$ is piecewise constant on interval $] \ n, n+1 \ [\ , \ \forall \ n \in Z \ \}$.

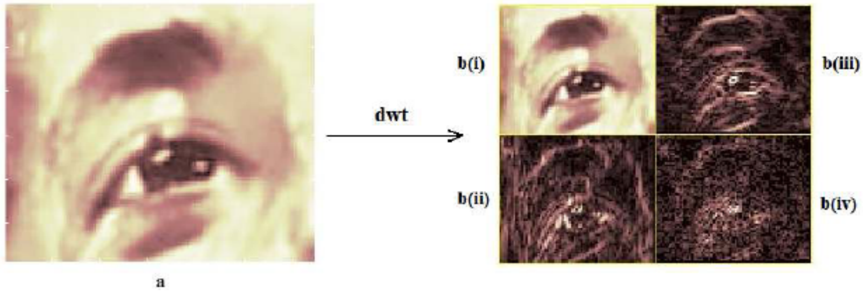

Figure 4: 2D example, showing image 0.5 resolution and incremental information

The subscript '0' is used for V_0 because of piecewise constancy on interval of size 2^{-0}.
Similarly, V_1 is defined as,
V_1 : { $x(t)$, such that $x(\cdot) \in L_2(\mathbb{R})$ is piecewise constant on all $]2^{-1}n, 2^{-1}(n+1)[$, $\forall\ n \in Z$ }.
V_1 is the set of functions piecewise constant over the interval of 2^{-1}.
In general V_m is the set of functions which is piecewise constant over the interval of size 2^{-m}.
V_m : { $x(t)$, such that $x(\cdot) \in L_2(\mathbb{R})$ is piecewise constant on all $]2^{-m}n, 2^{-m}(n+1)[$, $\forall\, n \in Z$}.

Example of a function $x(\cdot) \in V_2$

All function belonging to V_2 are piecewise constant over the interval of 0.25. Figure 5 shows a function belonging to V_2. Here $x(t) \in V_2$ means $x(t) \in L_2(\mathbb{R})$. This implies that the squared sum of all the piecewise constant values must be convergent.

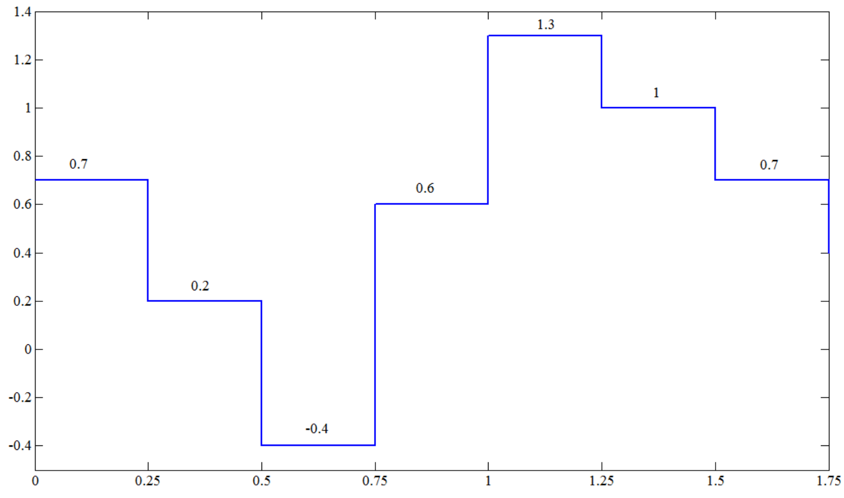

Figure 5: Example of a function belonging to V_2 space

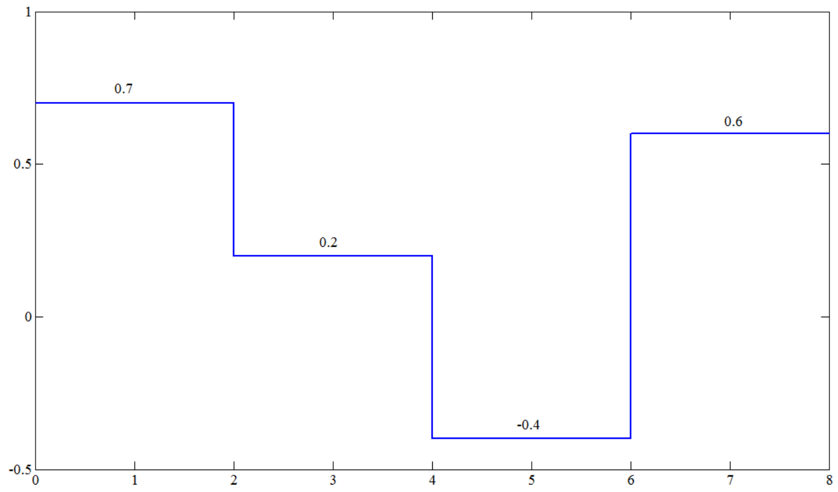

Figure 6: Example of a function belonging to V_{-1} space

Example of function $x(\cdot) \in V_{-1}$

Any function belonging to V_{-1} is piecewise constant over the interval of length two. Figure 6 shows a function belonging to V_{-1}. Now a function which is piecewise constant over the interval of 1 is also piecewise constant on the interval of 0.5. Therefore, a function belonging to space V_0 also belongs to space V_1. In general a function which belongs to space V_m also belongs to space V_{m+1}. Hence a ladder of subspaces is implied.

$$...V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2...$$

Intuitively we can see that as we move towards right i.e. up the ladder we are moving towards $L_2(\mathbb{R})$

$$\{\overline{\cup V_m}\}_{m\in\mathbb{Z}} = L_2(\mathbb{R})$$

Now what happens when we go left towards the ladder? Movement towards leftwards implies, piecewise constant approximation over larger and larger intervals. Now consider L_2 norm of function going towards leftwards:

$$\sum_{n=-\infty}^{\infty} |C_m(n)|^2 2^{-m}$$

where $C(\cdot)$ is approximate coefficient at resolution 2^{-m}. Now as we move towards left m becomes negative and $m \to -\infty$. Therefore L_2 norm is given by

$$2^{|m|} \sum_{n=-\infty}^{\infty} |C_m(n)|^2$$

If we require L_2 norm to converge, however for large $|m|$, $\sum_{n=-\infty}^{\infty} |C_m(n)|^2$ must be zero. That is $C_m = 0 \; \forall \; n$. Hence movement towards left implies movement towards trivial subspace $\{0\}$.

$$\{\cap V_m\}_{m\in\mathbb{Z}} = \{0\}$$

We say that a set of functions $\{f_1, f_2, f_3, ..., f_k, ...\}$ span a whole space if any function in that space can be represented by linear combination of these functions.
What is function $\phi(t)$ and how does its integer translates span V_0? We may consider function $\phi(t)$ as shown in figure 7.

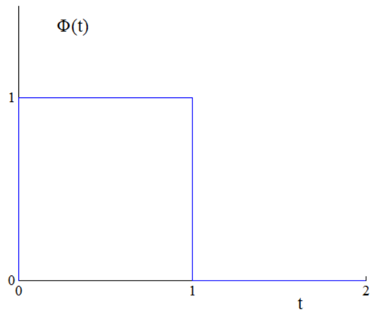

Figure 7: Function $\phi(t)$

Any function in V_0 can be expressed in the form

$$\sum_{n\in\mathbb{Z}} C_n \phi(t-n)$$

where C_n is piecewise approximation constants and $\phi(t-n)$ are integer translates of the $\phi(t)$. Figure 8 shows a function belonging to V_0. It can be expressed as shown below.

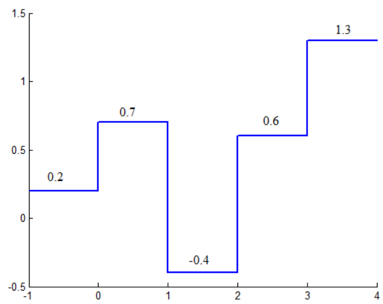

Figure 8: Example of a function belonging to V_0

$$0.2\phi(t+1) + 0.7\phi(t) - 0.4\phi(t-1) + 0.6\phi(t-2) + 1.3\phi(t-3)$$

Hence any space V can be similarly constructed using a function $\phi(2^m t)$.

$$V_m = span\{\phi(2^m t - n)\}_{n,m\in\mathbb{Z}}$$

$\phi(t)$ is called as scaling function(Haar MRA), which is also called as 'Father function'. The ladder of subspaces

$$...V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2...$$

with these properties is called as Multi-Resolution Analysis(MRA).

2 Axioms of MRA

There exists a ladder of subspaces, $\quad ...V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2... \quad$ such that

1. $\{\overline{\cup V_m}\}_{m\in\mathbb{Z}} = L_2(\mathbb{R})$
2. $\{\cap V_m\}_{m\in\mathbb{Z}} \ = \ \{0\}$
3. There exists $\phi(t)$ such that, $\quad V_0 = span\{\phi(t-n)\}_{n\in\mathbb{Z}}$
4. $\{\phi(t-n)\}_{n\in\mathbb{Z}}$ is an orthogonal set.
5. If $\quad f(t) \in V_m \quad then \quad f(2^{-m}t) \in V_0, \qquad \forall m \in \mathbb{Z}$
6. If $\quad f(t) \in V_0 \quad then \quad f(t-n) \in V_0, \qquad \forall n \in \mathbb{Z}$

3 Theorem of MRA

Given the axioms, there exists a $\quad \psi(\cdot) \in L_2(\mathbb{R})$, so that $\quad \{\psi(2^m t-n)\}_{m\in\mathbb{Z},n\in\mathbb{Z}} \quad$ spans the $L_2(\mathbb{R})$.

The wavelet function $\psi(\cdot)$ is also called as 'Mother function'.