WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 2: Haar Multiresolution analysis

Prof.V. M. Gadre, EE, IIT Bombay

1 Introduction

HAAR was a mathematician, who has given an idea that any continuous function can be represented in the form of discontinuous functions, and by doing so one can go to any level of continuity that one desires. That is, we start from a very discontinuous function and make it smoother and smoother by adding more and more discontinuous functions (additional information) to it. This idea is opposite to the idea of Fourier transform. As in Fourier transform the discontinuous function is represented in the form of smooth continuous function. Representation of the continuous function in the form of discontinuous function has its own importance in digital communication because in digital signal processing we are doing the same by converting smooth signal into a stream of bits (discontinuous function). This is illustrated by some examples.

Example 1

In a digital camera, the image is divided into small elements (called pixel), now consider an image of dimension 204 X 92 pixels. In each pixel, area of pixel is represented by a constant number which represents the average intensity and colour in that area. The effect of changing the resolution of the same image is seen in figure 1.

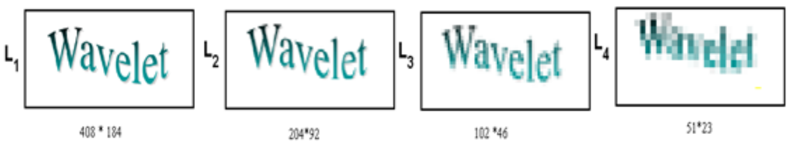

Figure 1: Resolution difference

Example 2

Consider an audio output. It is one dimensional signal which can be plotted against time. Let the output be as shown in figure 2. Audio signal is divided in small intervals of time 'T'. Lets represent each interval with a piecewise constant approximation. A piece wise constant approximation C_0 over the open interval (0,T) can be computed as

$$C_0 = \frac{1}{T}\int_0^T x(t)\ dt$$

(Open interval is the interval excluding end points). And for any interval of size T is given by

$$C_n = \frac{1}{T}\int_T x(t)\ dt$$

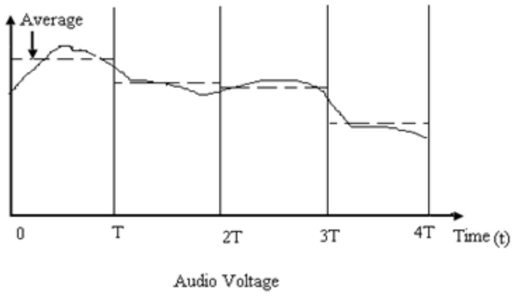

Figure 2: audio signal

i.e. integration over the time interval T. Similarly, for interval of size T/2,

$$C_{n,T/2} = \frac{1}{T/2} \int_{\frac{T}{2}} x(t)\ dt$$

If a time interval of length 'T' is divided into two time intervals of length T/2, we get two averages computed by the above formula. The figure 3 clarifies this concept further.

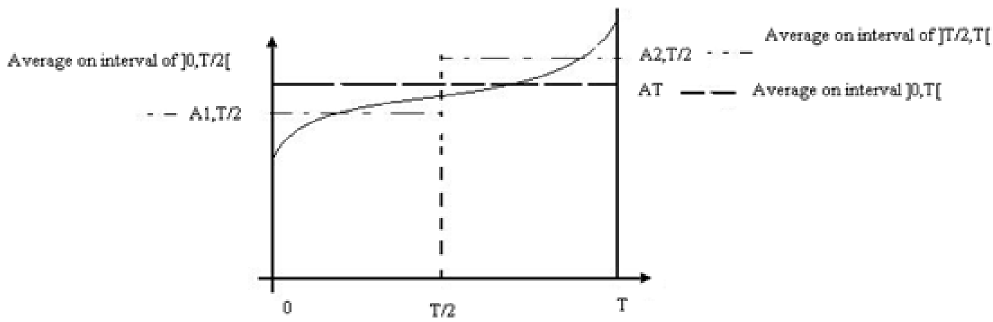

Figure 3: Average of T and T/2

$$\begin{aligned} A_T &= \frac{1}{T} \int_0^T x(t)\ dt \\ A_{1,T/2} &= \frac{2}{T} \int_0^{T/2} x(t)\ dt \\ A_{2,T/2} &= \frac{2}{T} \int_{T/2}^{T} x(t)\ dt \end{aligned}$$

The central concept in HAAR multi resolution analysis is to relate these three terms (A_T, $A_{1,T/2}$, $A_{2,T/2}$). The HAAR wavelet is hidden in this relationship. It can be easily observed that, the average of $A_{1,T/2}$ and $A_{2,T/2}$ gives A_T, *i.e.*

$$A_T = \frac{A_{1,T/2} + A_{2,T/2}}{2}$$

Thus a function can be approximately represented by addition of **piecewise constant** fucntions. We can go on reducing the interval by half to whatever degree of accuracy we desire. This is illustrated in the figure 4. Each different line (dot dash, dash-dash, bold line with

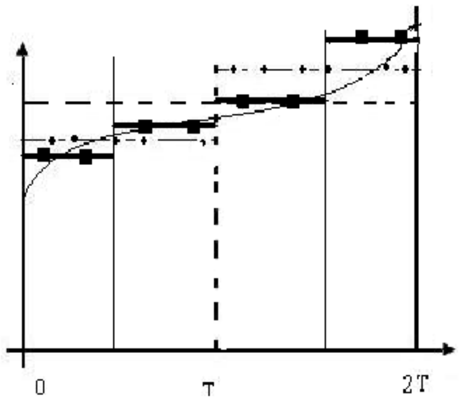

Figure 4: intervals of 2T,T and T/2

squares) represents a piecewise approximation in its own way, with different resolution. Let the function represented in above figure using dot-dash line be $f_1(t)$ and the bold line with squares be $f_2(t)$ then, the additional information obtained by representing the signal as $f_2(t)$ is given by

$$f_2(t) - f_1(t)$$

It is shown in figure 5.

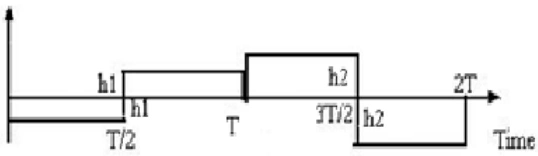

Figure 5: $f_2(t) - f_1(t)$

2 The Haar wavelet

Consider the following function shown in the figure 6. This function is represented as $\psi(t)$. By using scalar multiplication and delaying, we can see that $f_2(t) - f_1(t)$ can be reconstructed from $\psi(t)$. Thus,

$$f_2(t) - f_1(t) \;=\; -h_1 \times \psi(t/T) + h_2 \times \psi(\frac{t-T}{T})$$

The function $\psi(t)$ is called the **Haar wavelet**. In general when we start with $\psi(t)$ we can construct a function $\psi(\frac{t-\tau}{s})$ as a building block, where 's' is positive real and τ should be real. The variable 's' dilates $\psi(t)$ and 'τ' translates $\psi(t)$. The variable τ is called the **translation**

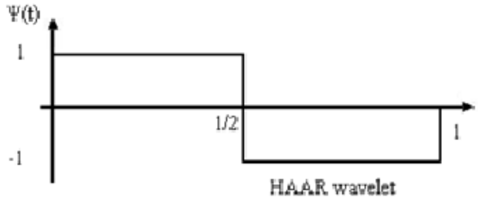

Figure 6: $f_2(t) - f_1(t)$

index and the variable 's' is called the **dilation variable**. If we consider time intervals of length T/2 for piecewise constant approximation then the value of 's' is T/2, if length of time interval is T then s=T. It means the single function $\psi(t)$ allows you to bring in resolution step by step to any level of detail. Thus, by dividing T into smaller subdivisions of T/2, T/4 and so on, any function $x_a(t)$ can be made **arbitrarily close** to original function $x(t)$. If $x_e(t)$ denotes the error due to approximation, it can be expressed as

$$x_e(t) = x(t) - x_a(t)$$
$$\zeta = \int_{-\infty}^{\infty} | x_e(t) |^2 \, dt$$

Where, ζ is the squared error. What we mean by arbitrarily close is that for any fixed value of $\zeta (> 0)$, we can always find a positive integer **m** such that a piecewise constant approximation of $x(t)$ with an interval of $T/2^m$ satisfies the requirement of ζ.

Important: Signal can be represented in piece wise constant form **if and only if** it has **finite energy**.

3 L_k norms of $x(t)$

A function have finite energy content implies and is implied by its L_2 norm being finite. The L_2 norm of a signal is defined as

$$L_2 \; norm \; of \; x(t) = \left[\int_{-\infty}^{\infty} | x(t) |^2 \, dt \right]^{\frac{1}{2}}$$

In general, we can define the L_p norm of x(t) as

$$L_p \; norm \; of \; x(t) = \left[\int_{-\infty}^{\infty} | x(t) |^p \, dt \right]^{\frac{1}{p}}$$

where p is any real number. The L_∞ norm of $x(t)$ is defined as

$$L_\infty \; norm \; of \; x(t) = \lim_{p \to \infty} \left[\int_{-\infty}^{\infty} | x(t) |^p \, dt \right]^{\frac{1}{p}}$$

Significance of L_∞ norm

As the value of p increases, large values in $x(t)$ are being emphasized. This happens because for a large p, the integral will have a large contribution from higher values in $x(t)$.

Space L_2

$L_2(\mathbb{R})$ is said the to be the space of all **real** functions whose L_2 norm is finite.