NPTEL COURSE

SELECTED TOPICS IN MATHEMATICAL PHYSICS

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QUIZ 1

1. Are the statements in quotation marks true or false?

   (a) "Every derivative of an analytic function of a complex variable is also an analytic function."

   (b) Let u and v denote the real and imaginary parts of an analytic function of $z = x + iy$.
   "The curves $u(x, y) =$ constant and $v(x, y) =$ constant intersect each other at right angles."

   (c) "An entire function must necessarily be singular at $z = \infty$, unless it is just a constant."

   (d) "A meromorphic function cannot have an essential singularity at the point at infinity."

   (e) "The radius of convergence of the power series $\sum_{n=1}^{\infty} n^{1/n}\, z^n$ is zero."

   (f) "The function $\sin(\pi/z)$ has an accumulation point of poles at $z = 0$."

   (g) "The relation $\Gamma(z)\,\Gamma(1-z) = \pi \operatorname{cosec} \pi z$ is only valid in the region $0 < \operatorname{Re} z < 1$."

   (h) Let a be a positive constant.
   "$f(z) = \int_a^\infty dt\, t^z\, e^{-t}$ is an entire function of z."

   (i) "The power series $\sum_{n=1}^{\infty} z^n/n^4$ is absolutely convergent at all points inside and on the unit circle $|z| = 1$."

(j) "The series $\sum_{n=0}^{\infty}(n+1)^{z-1}$ converges in the region $\mathrm{Re}\, z > 0$."

(k) Consider the Möbius transfomation $z \mapsto w = (2z + \sqrt{3}\,)/(\sqrt{3}\,z + 2)$.
"This is a hyperbolic Möbius transformation."

(l) Consider $z \mapsto w = (2z + \sqrt{3}\,)/(\sqrt{3}\,z + 2)$ once again.
"The transformation maps the circle $|z| = 1$ to the circle $|w| = 1$."

(m) Legendre's differential equation is $(1 - z^2)\phi'' - 2z\phi' + \nu(\nu + 1)\phi = 0$.
"Since this equation is invariant under the interchange $\nu \leftrightarrow -\nu - 1$, *all* its solutions must also be invariant under this interchange."

(n) "The function $f(t) = e^{t^{3/2}}$, where $t \geq 0$, has no Laplace transform."

(o) "If $[\mathcal{L}f](s) = \int_0^\infty dt\, e^{-st}\, f(t)$, then $[\mathcal{L}^2 f](s) = \int_0^\infty dt\, f(t)/(s + t)$."

(p) "The product $\Gamma(z)\,\zeta(z)$ tends to a finite, nonzero limit as $z \to -2n$, where $n = 1, 2, \ldots$."

(q) Let $\phi(t)$ be a linear, causal, retarded response function, and $\widetilde{\phi}(s)$ its Laplace transform.
"The corresponding dynamic susceptibility $\chi(\omega)$ is the analytic continuation of $\widetilde{\phi}(s)$ to $s = -i\omega$."

(r) "The logarithmic derivative of the Riemann zeta function, $\zeta'(z)/\zeta(z)$, has a simple pole at $z = 1$ with residue equal to -1."

(s) "The only pole of the logarithmic derivative of the Riemann zeta function is at $z = 1$."

(t) Consider the function space $\mathcal{L}_2(-\infty\,,\,\infty)$.
"Any eigenfunction of the Fourier transform operator is also an eigenfunction of the parity operator, but the converse is not necessarily true."

2. Fill in the blanks in the following.

(a) The real part of an entire function $f(z)$ is given by $u(x, y) = (\cosh\, x)\,(\cos\, y)$. Hence the function is $f(z) \;= \cdots$.

(b) $\coth z$ is a periodic function of z, with a period equal to $\cdots$.

(c) The singularity of the polynomial $p(z) = z^n + a_1\, z^{n-1} + \cdots + a_n$ at $z = \infty$ is $\cdots$. (Select one from the following: (i) a removable singularity (ii) a simple pole (iii) a pole of order n (iv) an essential singularity.)

(d) The residue at $z = \infty$ of the polynomial $p(z) = z^n + a_1\, z^{n-1} + \cdots + a_n$ is $\operatorname*{Res}_{z=\infty} p(z) = \;\cdots$.

(e) Let C denote the circle $|z| = 2$ traversed once in the positive sense. Then $\oint_C dz/(z^4 - 1) = \;\cdots$.

(f) Let a and b be two different complex numbers, each with nonzero real and imaginary parts. The radius of convergence of the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+a)}{\Gamma(a)}\, \frac{\Gamma(b)}{\Gamma(n+b)}\, \frac{z^n}{n!}$$

is $R = \;\cdots$.

(g) Given that $\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90$, it follows that $\sum_{n=0}^{\infty} 1/(2n+1)^4 = \;\cdots$.

(h) Given that $\int_0^\infty dx\,(\sin\, kx)/x = \frac{1}{2}\pi$ (where $k > 0$), the value of the integral $\int_0^\infty dx\,(1 - \cos\, x)/x^2 = \;\cdots$. (*Hint*: Integrate k over a suitable range.)

(i) The numerical value of the product

$$\Gamma\big(-\tfrac{5}{4}\big)\,\Gamma\big(-\tfrac{3}{4}\big)\,\Gamma\big(-\tfrac{1}{4}\big)\,\Gamma\big(\tfrac{1}{4}\big)\,\Gamma\big(\tfrac{3}{4}\big)\,\Gamma\big(\tfrac{5}{4}\big)\,\Gamma\big(\tfrac{7}{4}\big)\,\Gamma\big(\tfrac{9}{4}\big) = \;\cdots.$$

(j) The value of the integral $\int_0^1 dt\, t^{-1/2}\,(1-t)^{-1/2} = \;\cdots$.

(k) Let an arbitrary initial point $z^{(0)}$ in the complex plane be mapped to the point $z^{(n)}$ under n iterations of the Möbius transfomation

$$z \mapsto (2z + \sqrt{3}\,)/(\sqrt{3}\, z + 2).$$

As $n \to \infty$, $z^{(n)} \to \cdots$ for all $z^{(0)}$, with *one* exception.

(l) Previous question continued: The exceptional point that does *not* tend to the limit point above is $z = \cdots$.

(m) Under the Möbius transformation $z \mapsto w = (2z+3)/(z+2)$, the circle $|z+2| = 1$ is mapped to the circle $\cdots$.

(n) The function $f(z) = \sqrt{z} \ln[(z-1)/(z+1)]$ has branch points at $z = \cdots$.

(o) Let α and β be arbitrary complex numbers. The function

$$f(z) = (z^2 - 1)^\alpha / (z^2 + 1)^\beta$$

has branch points at $z = \cdots$.

(p) The residue of $f(z) = \exp(z + z^{-1})$ at $z = 0$ is $\cdots$. (Express your answer in terms of a modified Bessel function.)

(q) Given that the Laplace transform of $\sin t$ is $1/(s^2+1)$, it follows that the Laplace transform of $\sinh t$ is $\cdots$.

(r) The generating function for the Hermite polynomial $H_n(z)$ is

$$e^{2tz - t^2} = \sum_{n=0}^{\infty} H_n(z) \frac{t^n}{n!}.$$

It follows that the Rodrigues formula for $H_n(z)$ is $H_n(z) = \cdots$.

(s) Let

$$f(x) = \begin{cases} 1, & |x| \le 1 \\ 0 & |x| > 1. \end{cases}$$

If $\widetilde{f}(k)$ denotes the Fourier transform of $f(x)$, the value of the integral $\int_{-\infty}^{\infty} dk \, |\widetilde{f}(k)|^2 = \cdots$.

(t) Consider a random walk on an infinite linear lattice whose sites are labelled by the integers. The walker jumps from any site j to $j-1$ with a probability per unit time given by λq, and from j to $j+1$ with a probability per unit time given by λp; further, the walker stays at the site j with probability per unit time given by λr. Here p, q and r are positive constants satisfying $p + q + r = 1$, and λ is a positive constant with the physical dimensions of $(\text{time})^{-1}$. Let $P(j, t)$ be the probability that the walker is at the site j at time t. The differential equation satisfied by $P(j, t)$ is $dP(j, t)/dt = \cdots$.

Quiz 1: Solutions

1. True or false:

   (a) **T**
   (b) **T**
   (c) **T**
   (d) **F**
   (e) **F**
   (f) **F**
   (g) **F**
   (h) **T**
   (i) **T**
   (j) **F**
   (k) **T**
   (l) **T**
   (m) **F**
   (n) **T**
   (o) **T**
   (p) **T**
   (q) **T**
   (r) **T**
   (s) **F**
   (t) **T**

2. Fill in the blanks:

   (a) $f(z) = \cosh z$
   (b) $i\pi$
   (c) a pole of order n
   (d) 0
   (e) 0
   (f) ∞
   (g) $\pi^4/96$

(h) $\frac{1}{2}\pi$

(i) $4\pi^4$

(j) π

(k) 1

(l) -1

(m) $|w-2| = 1$

(n) $-1, 0, 1$ and ∞

(o) $1, i, -1, -i$ and ∞

(p) $I_1(2)$

(q) $1/(s^2-1)$

(r) $H_n(z) = \left[\dfrac{d^n}{dt^n} e^{2tz-t^2}\right]_{t=0}$, which simplifies to $H_n(z) = (-1)^n\, e^{z^2}\, \dfrac{d^n}{dz^n} e^{-z^2}$.

(s) 4π

(t) $\dfrac{dP(j,t)}{dt} = \lambda\Big[p\,P(j-1,t) + q\,P(j+1,t) - (p+q)\;P(j,t)\Big]$

QUIZ 2

1. Are the statements in quotation marks true or false?

   (a) "The function $\sin(1/z)$ does not have a Taylor series expansion in the neighborhood of $z = 0$."

   (b) A function $f(z)$ is defined by the power series $\sum_{n=0}^{\infty} z^{2n+1}/[n!\,(n+1)!]$ about the origin.

   "$f(z)$ is an entire function of z."

   (c) "The only singularity of $1/\Gamma(z)$ is a simple pole at $z = 0$."

   (d) "The Mittag-Leffler expansion of the gamma function is given by $\Gamma(z) = \sum_{n=0}^{\infty}(-1)^n/[(z+n)\,n!]$."

   (e) Let $f(z) = z + z^3 + z^9 + z^{27} + \cdots$ *ad infinitum*, for $|z| < 1$.

   "$f(z)$ cannot be analytically continued outside the unit circle."

   (f) "The power series $\sum_{n=1}^{\infty}(\ln n)\, z^n/n$ converges at all points on its circle of convergence."

   (g) "The function $f(z) = 1/(e^z - 1)$ is a meromorphic function of z."

   (h) "Dispersion relations for the real and imaginary parts of a generalized susceptibility $\chi(\omega)$ can be derived only if the corresponding response function $\phi(t)$ decays to zero faster than any negative power of t, as $t \to \infty$."

   (i) "The derivative of the gamma function, $\Gamma'(z)$, has zero residue at each of its poles."

   (j) "The Legendre function of the second kind, $Q_\nu(z)$, has branch points in the z-plane even when ν is a positive integer."

   (k) "The Laplace transform of the function $f(t) = \cosh \pi t$ has no singularities in the region $\mathrm{Re}\, s > \pi$."

(l) Bessel's differential equation is $\left[z^2 \dfrac{d^2}{dz^2} + z\, \dfrac{d}{dz} + (z^2 - \nu^2)\right] f(z) = 0$, where ν is a parameter.

"If $\phi_\nu(z)$ is any solution of this equation, then $\phi_{-\nu}(z)$ must be equal to $\phi_\nu(z)$, apart from a possible multiplicative constant."

(m) "The group $\text{Möb}\,(2, \mathbb{C})$ of Möbius transformations of the complex plane has continuous subgroups, but no discrete subgroups."

(n) "The group $\text{Möb}\,(2, \mathbb{C})$ of Möbius transformations of the complex plane is isomorphic to the group $SO(3, 1)$ of homogeneous proper Lorentz transform-stions in $(3 + 1)$-dimensional spacetime."

(o) "The Riemann surface of the function $f(z) = z^{1/2}\,(z - 1)^{-1/3}$ has 6 sheets."

(p) "It is possible to find a contour integral representation of the beta function $B(z, w)$ that is valid for all complex values of both z and w."

(q) "The Riemann zeta function $\zeta(z)$ cannot be continued analytically to the left of the line $\text{Re}\, z = \frac{1}{2}$, because it has an infinite number of zeroes on that line."

(r) "The Fourier transform operator in $\mathcal{L}_2(-\infty\,, \infty)$ has a finite number of eigenvalues, each of which is infinitely degenerate."

(s) Let $G(x, x')$ denote the Green function of the differential operator d^2/dx^2 where $x \in [-1, 1]$.

"As a function of x, G is continuous at $x = x'$, but its derivative $\partial G/\partial x$ has a finite discontinuity at $x = x'$."

(t) "The fundamental Green function of the Laplacian operator ∇^2 in four-dimensional Euclidean space is $G(\mathbf{r}, \mathbf{r}') = -1/[4\pi^2(\mathbf{r} - \mathbf{r}')^2]$."

(u) Consider the diffusion equation in d-dimensional space, $\partial f/\partial t = D\nabla^2 f$ with boundary condition $f(\mathbf{r}, t) \to 0$ as $r \to \infty$ and initial condition $f(\mathbf{r}, 0) = \delta^{(d)}(\mathbf{r})$.

"The fundamental solution to this equation is a Gaussian in each Cartesian component of $\mathbf{r}$, for all positive integer values of the dimension d."

(v) The scattering amplitude for the scattering of a nonrelativistic particle of mass m in a central potential $\lambda V(r)$ is given by

$$f(k,\theta) = -\frac{m\lambda}{2\pi\hbar^2}\int d^3r\, e^{-i\mathbf{k}'\cdot\mathbf{r}}\, V(r)\,\psi(\mathbf{r}),$$

where $\mathbf{k}'$ is the scattered wave vector.

"This formula is valid only if the potential $V(r)$ decays to zero as $r \to \infty$ more rapidly than any inverse power of r."

(w) Continuation: "In the Born approximation, the scattering amplitude in the forward direction ($\theta = 0$) vanishes identically."

(x) Continuation: "In the Born approximation, the imaginary part of the scattering amplitude vanishes identically."

(y) Consider the Helmholtz operator $\nabla^2 + \mathbf{k}^2$ in three-dimensional space.

"The fundamental Green function of this operator, corresponding to outgoing spherical waves, is $G(\mathbf{r}-\mathbf{r}') = -e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}/\big(4\pi|\mathbf{r}-\mathbf{r}'|\big)$."

(z) Consider the wave operator $(1/c^2)\,\partial^2/\partial t^2 - \nabla^2$ in $(d+1)$-dimensional spacetime, where c is the speed of light in a vacuum. Let $G^{(d)}(R,\tau)$ denote the causal retarded Green function of the operator.

"$G^{(d)}(R,\tau)$ vanishes identically when $(c\tau, \mathbf{R})$ is a time-like four-vector."

(α) Continuation: "$G^{(d)}(R,\tau)$ is singular when $(c\tau, \mathbf{R})$ is a light-like four-vector."

(β) Let $\mathbf{J} = (J_i\,,\, J_2\,,\, J_3)$ be the generators of rotations in three-dimensional space, satisfying the Lie algebra $[J_j\,,\, J_k] = i\epsilon_{jkl}\, J_l\,$.

"The lowest-dimensional, non-trivial, unitary representation of the generators is in terms of (2×2) matrices with complex elements."

(γ) "The parameter space of the group $SU(n)$ is doubly connected."

(δ) "The first homotopy group of the parameter space of the special orthogonal group $SO(n)$, for every $n \geq 3$, is $\mathbb{Z}_2$."

2. Fill in the blanks in the following.

   (a) Given that the imaginary part of an entire function $f(z)$ is

   $$v(x, y) = e^{(x^2-y^2)} \sin(2xy),$$

   the function is $f(z) = \cdots$.

   (b) The meridian of longitude φ on the Riemann sphere is mapped into a straight line in the complex plane. The equation of this straight line is $y = mx + c$, where $m = \cdots$ and $c = \cdots$.

   (c) The region of absolute convergence in the complex z-plane of the power series $\sum_{n=0}^{\infty} \big[(n+1)/(n^2+1)\big]\big(\frac{1}{2}z\big)^n$ is $\cdots$.

   (d) The residue at infinity of the function $f(z) = (z - z^{-1})^3$ is $\operatorname*{Res}_{z=\infty} f(z) = \cdots$.

   (e) Let $[z_1, z_2\,;\, z_3, z_4]$ denote the cross-ratio of the four points z_1, z_2, z_3 and z_4 in the complex plane. Then $[z_1, z_2\,;\, z_3, z_4] + [z_1, z_3\,;\, z_2, z_4] = \cdots$.

   (f) The Möbius transformation $z \mapsto w$ such that three given points z_1, z_2, z_3 are mapped respectively into three other given points w_1, w_2, w_3 is expressed by a relation between w and z that reads $\cdots$.

   (g) Under the Möbius transformation $z \mapsto w = (z+1)/(z+2)$, an infinitesimal area element δA centered at the point $z = -3/2$ is mapped to an element of area $\lambda\,\delta A$, where the value of λ is $\cdots$.

   (h) The Bernoulli numbers B_n are defined via the expansion $z/(e^z - 1) = \sum_{n=0}^{\infty} B_n\, z^n/n!$. Therefore B_n is given by the contour integral $B_n = \cdots$. (You must specify both the integrand and the contour.)

   (i) The Chebyshev polynomial of the second kind, $U_n(\cos\theta)$, has the generating function

   $$\frac{1}{1 - 2t\cos\theta + t^2} = \sum_{n=0}^{\infty} U_n(\cos\theta)\, t^n,$$

   where $\theta \in [0, \pi]$. Therefore $U_n(\cos\theta)$ can be expressed as a contour integral in the t-plane given by $U_n(\cos\theta) = \cdots$. (You must specify both the integrand and the contour.)

   (j) Continuation: Evaluating the contour integral and simplifying the result, the final expression for $U_n(\cos\theta)$ is $U_n(\cos\theta) = \cdots$. (You must express your answer in terms of trigonometric functions of θ.)

   (k) Continuation: Hence the polynomial $U_1(\cos\theta)$ reduces to $U_1(\cos\theta) = \cdots$.

   (l) The function $f(z) = (z^2 + 2)^{1/3}$ has branch points at $z = \cdots$.

(m) Express your answer in terms of a Bessel function:
The residue of $f(z) = \exp(z - z^{-1})$ at $z = 0$ is $\operatorname*{Res}_{z=0} f(z) = \cdots$.

(n) The inverse Laplace transform of $\widetilde{f}(s) = 1/(s^2 - 2s + 1)$ is $f(t) = \cdots$.

(o) Let λ be a positive constant. The Laplace transform of the function

$$f(t) = \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 \, e^{-\lambda(t-t_1)}$$

is $\widetilde{f}(s) = \cdots$.

(p) Let

$$f(x) = \begin{cases} 1 - |x|, & |x| \le 1 \\ 0, & |x| > 1. \end{cases}$$

If $\widetilde{f}(k)$ denotes the Fourier transform of $f(x)$, the value of the integral

$$\int_{-\infty}^{\infty} dk \, |\widetilde{f}(k)|^2 = \cdots .$$

(q) The positional probability distribution at any time $t \ge 0$ of a random walker on a square lattice with sites labelled by the integers (ℓ, m) is given by

$$P(\ell, m, t) = e^{-\lambda t} \, (p_1/q_1)^{\ell/2} \, (p_2/q_2)^{m/2} \, I_\ell\big(2\lambda t \sqrt{p_1 \, q_1}\,\big) \, I_m\big(2\lambda t \sqrt{p_2 \, q_2}\,\big),$$

where λ is the mean jump rate and p_i, q_i are directional probabilities such that $p_1 + q_1 + p_2 + q_2 = 1$. The leading asymptotic behavior of $P(\ell, m, t)$ at very long times ($\lambda t \gg 1$) is given by $P(\ell, m, t) \sim \cdots$.

(r) The diffusion equation for the positional probability density of a particle diffusing on the x-axis in the region $-\infty < x \le a$, in the presence of a constant force, is given by

$$\frac{\partial p(x,t)}{\partial t} = -c \, \frac{\partial p\,(x,t)}{\partial x} + D \, \frac{\partial^2 p\,(x,t)}{\partial x^2}.$$

Here c and D are positive constants denoting the drift velocity and diffusion constant, respectively. $p(x,t)$ is normalized to unity for all $t \ge 0$. There is a reflecting boundary at the point $x = a$. The boundary condition satisfied by $p(x,t)$ at $x = a$ is then given by $\cdots$.

(s) Continuation: As $t \to \infty$, $p(x,t)$ tends to the stationary probability density $p_{\mathrm{st}}(x)$. This quantity satisfies the ordinary differential equation $\cdots$.

(t) Continuation: The normalized solution for $p_{\mathrm{st}}(x)$ is $p_{\mathrm{st}}(x) = \cdots$.

(u) A quantum mechanical particle of mass m moving in one dimension has the Hamiltonian $H = p^2/(2m)$, where p is the momentum operator of the particle. Its momentum-space wave function at $t = 0$ is given to be $\phi(p)$. Therefore its momentum-space wave function at any time $t \ge 0$ is given by $\psi(p,t) = \cdots$.

(v) The scattering amplitude for a nonrelativistic particle of mass m in a central potential $\lambda V(r)$ is given, in the Born approximation, by

$$f_{\rm B}(k,\theta) = -\frac{2m\lambda}{\hbar^2\,Q}\int_0^\infty dr\, r\,\sin{(Qr)}\,V(r),$$

where Q is the magnitude of the momentum transfer vector $\mathbf{Q}$. The forward scattering amplitude in the Born approximation is therefore given by the expression $f_{\rm B}(k,0) = \cdots$.

(w) Continuation: The backward scattering amplitude in the Born approximation is therefore given by the expression $f_{\rm B}(k,\pi) = \cdots$.

(x) Let $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $\tau = t - t'$, as usual. Let $G^{(d)}(\mathbf{R},\tau)$ denote the fundamental Green function of the Klein-Gordon operator $\Box + \mu^2$, where μ is a positive constant and $\Box = (1/c^2)(\partial^2/\partial t^2) - \nabla^2$, in $(d+1)$-dimensional spacetime. Then $G^{(d)}(\mathbf{R},\tau)$ can be expressed in the form

$$G^{(d)}(\mathbf{R},\tau) = \frac{1}{(2\pi)^d}\int d^d\mathbf{k}\;\phi(\mathbf{k},\mathbf{R},\tau),$$

where $\phi(\mathbf{k},\mathbf{R},\tau) = \cdots$.

(y) Let $\mathbf{J} = (J_1\,,\,J_2\,,\,J_3)$ denote the generators of rotations in three-dimensional space, and let ψ be an arbitrary angle. The quantity $e^{-iJ_1\,\psi}\,J_2\,e^{iJ_1\,\psi}$, expressed as a linear combination of the generators, is $\cdots$.

(z) Continuation: Let $\mathbf{n} = (n_1\,,\,n_2\,,\,n_3)$ be an arbitrary unit vector. Then the commutator $[\mathbf{J}\cdot\mathbf{n}\,,\,\mathbf{J}^2] =$.

(α) The number of generators of the orthogonal group $O(n)$ and the special orthogonal group $SO(n)$ are, respectively, $\cdots$ and $\cdots$.

(β) The number of generators of the unitary group $U(n)$ and the special unitary group $SU(n)$ are, respectively, $\cdots$ and $\cdots$.

(γ) Let x and p denote the position and momentum operators of a quantum mechanical particle moving in one dimension, so that their commutator $[x\,,\,p] = i\hbar I$, where I is the unit operator. Let a be a real constant with the physical dimensions of length. Using Hadamard's Lemma, the operator $e^{iap/\hbar}\,x\,e^{-iap/\hbar}$ simplifies to

$$e^{iap/\hbar}\,x\,e^{-iap/\hbar} = \cdots\,.$$

(δ) Continuation: Let b be a real constant with the physical dimensions of linear momentum. Once again, using Hadamard's Lemma, the operator $e^{-ibx/\hbar}\,p\,e^{ibx/\hbar}$ simplifies to

$$e^{-ibx/\hbar}\,p\,e^{ibx/\hbar} = \cdots\,.$$

Quiz 2: Solutions

1. True or false:

   (a) T
   (b) T
   (c) F
   (d) F
   (e) T
   (f) F
   (g) T
   (h) F
   (i) T
   (j) T
   (k) T
   (l) F
   (m) F
   (n) T
   (o) T
   (p) T
   (q) F
   (r) T
   (s) T
   (t) T
   (u) T
   (v) F
   (w) F
   (x) T
   (y) F
   (z) F

   (α) T
   (β) T
   (γ) F

(δ) T

2. Fill in the blanks:

   (a) $f(z) = e^{z^2}$.

   (b) $m = \tan\,\varphi$ and $c = 0$.

   (c) $|z| < 2$.

   (d) $\operatorname*{Res}_{z=\infty} f(z) = -3$.

   (e) 1.

   (f) $\dfrac{(w-w_2)(w_1-w_3)}{(w-w_3)(w_1-w_2)} = \dfrac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)}$.

   (g) $\lambda = 16$.

   (h) $B_n = \dfrac{n!}{2\pi i}\oint_C \dfrac{dz}{z^n\,(e^z-1)}$, where C encloses the origin once in the positive sense.

   (i) $U_n(\cos\,\theta) = \dfrac{1}{2\pi i}\oint_C \dfrac{dt}{t^{n+1}\,(1-2t\,\cos\,\theta+t^2)}$, where C encloses the origin once in the positive sense.

   (j) $U_n(\cos\,\theta) = \dfrac{\sin\,(n+1)\theta}{\sin\,\theta}$.

   (k) $U_1(\cos\,\theta) = 2\,\cos\,\theta$.

   (l) $i\sqrt{2},\ -i\sqrt{2},\ \infty$.

   (m) $-J_1(2)$.

   (n) $t\,e^t$.

   (o) $1/[s(s+\lambda)^n]$.

   (p) $4\pi/3$.

   (q) $P(\ell, m, t) \sim \dfrac{(p_1/q_1)^{\ell/2}\,(p_2/q_2)^{m/2}\,\exp\,\big[-\lambda t\,\big(1-2\sqrt{p_1\,q_1}-2\sqrt{p_2\,q_2}\big)\big]}{4\pi\lambda t\,(p_1\,q_1\,p_2\,q_2)^{1/4}}$.

   (r) $\Big[D\,\dfrac{\partial p}{\partial x} - c\,p\Big]_{x=a} = 0$.

   (s) $D\,\dfrac{d^2 p_{\rm st}(x)}{dx^2} - c\,\dfrac{dp_{\rm st}(x)}{dx} = 0$.

   (t) $p_{\rm st}(x) = (c/D)\,e^{c(x+a)/D}$.

   (u) $\psi(p,t) = e^{-ip^2t/(2m\hbar)}\,\phi(p)$.

   (v) $f_{\rm B}(k,0) = -\dfrac{2m\lambda}{\hbar^2}\displaystyle\int_0^\infty dr\,r^2\,V(r)$.

   (w) $f_{\rm B}(k,\pi) = -\dfrac{m\lambda}{\hbar^2\,k}\displaystyle\int_0^\infty dr\,r\,V(r)\,\sin\,(2kr)$.

(x) $\phi(\mathbf{k}, \mathbf{R}, \tau) = c\,\theta(\tau)\,\dfrac{\sin\, c\tau k}{k}\, e^{i\mathbf{k}\cdot\mathbf{R}}$.

(y) $J_2 \cos\psi + J_3 \sin\psi$.

(z) 0.

(α) $\frac{1}{2}n(n-1)$ and $\frac{1}{2}n(n-1)$.

(β) n^2 and $n^2 - 1$.

(γ) $x + a$. This result tells us why the momentum operator p is the generator of translations in position space.

(δ) $p + b$. This result tells us why the position operator x is the generator of translations in momentum space.