Department of Physics
Indian Institute of technology Madras
Select/Special Topics in Classical Mechanics
Self-Assessment-2 (Questions & Answers)

NOTE: Symbols/notations used in this question paper have their usual meanings, as used in our course.

☺ **SOLUTIONS** ☺

1. *State whether the following statements are 'TRUE' or 'FALSE'* and give reason. *The reason should be short, but as rigorous as you can provide.*

a. For a particle of mass m moves in a region of space where the potential is described by $U(x,y) = -U_0 \exp\left[-\frac{(x^2+y^2)}{2L^2}\right]$, the point (x=0, y=0) is a '*saddle point*' (given: U_0 & L are positive constants).

Solution: False

$$\frac{d^2U}{dx^2} = \frac{U_0}{L^2} \quad \textbf{at point } (x,y) = (0,0)$$

$$\frac{d^2U}{dy^2} = \frac{U_0}{L^2} \quad \textbf{at point } (x,y) = (0,0)$$

Since double derivative of the function with respect to the two variable is positive, the point at (x,y) = (0,0) is not a saddle point; it is a point of 'stable equilibrium' .

b. If a vector field $\vec{A}$ is both irrotational $(\vec{\nabla} \times \vec{A} = \vec{0})$ and solenoidal $(\vec{\nabla} \bullet \vec{A} = 0)$, then it must be identically equal to the *null vector*.

Solution: False

For any constant vector field $\vec{A}$, for example, $(\vec{\nabla} \times \vec{A} = \vec{0})$ and $(\vec{\nabla} \bullet \vec{A} = 0)$

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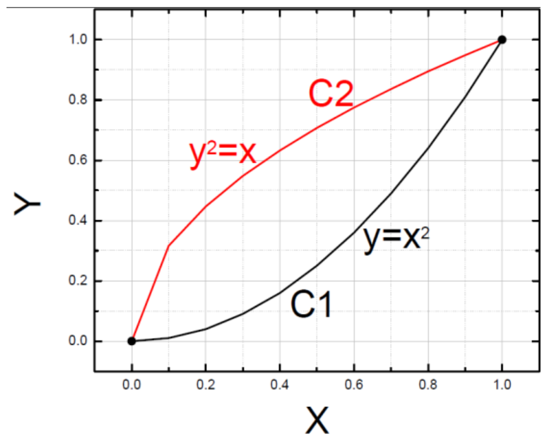
2. A position-dependent force field is given by the expression $\vec{F} = A(x-y)\hat{e}_x + (x+y)\hat{e}_y$. It is given that $|A| = +1$.



(a) What is/are the dimension(s) of A?

Solution:

Dimension of A must be $\frac{[\vec{F}]}{L} = MT^{-2}$

(b) The given force acts on a particle, moving it along a closed path described by the two curves:

$y = x^2$, *traversed from (0,0) to (1,1),*

and

$y^2 = x$ *traversed from (1,1) to (0,0).*

(c) Determine the work $\oint \vec{F} \bullet \vec{dl}$ done by the above force over the closed path described above.

Solution:

$$\vec{F} = A(x-y)\hat{e}_x + (x+y)\hat{e}_y \text{ with } |A|=1$$

$$\oint_C \vec{F} \bullet \vec{dl} = \int_{Along\ C_1} \vec{F} \bullet \vec{dl} + \int_{Along\ C_2} \vec{F} \bullet \vec{dl}$$

$$\int_{Along\ C_1} \vec{F} \bullet \vec{dl} = \int_0^1 (x-x^2)dx + (x+x^2)2xdx = \frac{4}{3}$$

$$\int_{Along\ C_2} \vec{F} \bullet \vec{dl} = \int_1^{01} (y^2-y)2ydy + (y^2+y)dy = -\frac{2}{3}$$

$$\therefore \oint_C \vec{F} \bullet \vec{dl} = \int_{Along\ C_1} \vec{F} \bullet \vec{dl} + \int_{Along\ C_2} \vec{F} \bullet \vec{dl} = \frac{2}{3}$$

(d) **<u>Without</u>** determining the curl of this force $\left(i.e. \text{ without finding } \vec{\nabla} \times \vec{F}\right)$, can you tell if the force is irrotational or not? **Explain how!**

Solution:

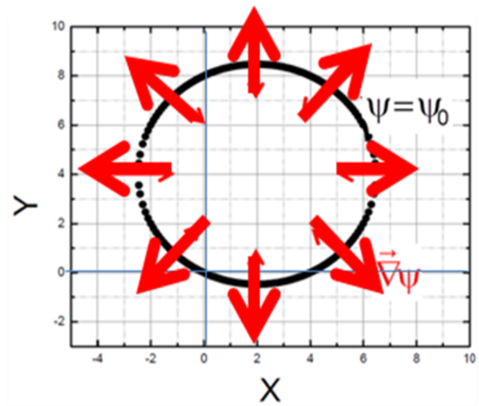
The Stokes' theorem states that $\oint \vec{F} \bullet \vec{dl} = \iint \left(\vec{\nabla} \times \vec{F}\right) \bullet \vec{ds}$. **In the present case, since** $\oint \vec{F} \bullet \vec{dl}$ **is nonzero,**

$\vec{\nabla} \times \vec{F}$ **must also be nonzero, which implies that** $\vec{F}$ **is not irrotational.**

3. A scalar field ψ(x, y) is given by the expression ψ(x, y) = ψ_0 $\exp(x^2+y^2-4x-8y)$,

where ψ_0 is a constant having suitable dimensions.

(a) Obtain the equipotential curve for $\psi = \psi_0$.



Solution:

$$\psi(x,\ y) = \psi_0 \ \exp(x^2+y^2-4x-8y)$$

$$\ln\frac{\psi}{\psi_0} = x^2+y^2-4x-8y$$

Add 20 to both sides,

$$\ln\frac{\psi}{\psi_0}+20 = x^2+y^2-4x-8y+20$$

$$=(x-2)^2+(y-4)^2$$

For constant value of $\psi=\psi_0$, the equipotential curve is a circle with centre at (x,y)=(2,4) and radius $\sqrt{20}\approx 4.47...$.

b) Sketch the vector field $\vec{\nabla}\psi$ at $\psi = \psi_0$.

Solution: $\vec{\nabla}\psi$ **is perpendicular to equipotential curves, as shown. Note that as** $\psi \geq \psi_0$**, the radius of the equipotential circle would be** $\geq \sqrt{20}$ **, so the gradient would be pointed OUTWARD.**

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4 (a) Determine the divergence of the vector point function described by:

$$\vec{A}(\hat{r})=(r\cos\theta)\,\hat{e}_r + (r\sin\theta)\,\hat{e}_\theta + (r\sin\theta\cos\varphi)\hat{e}_\phi$$

Solution:

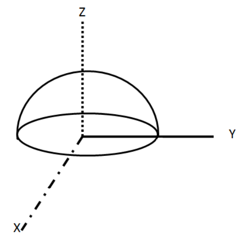
$$\vec{\nabla}\bullet\vec{A} = \left\{\hat{e}_r\frac{\partial}{\partial r}+\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right\}\bullet\left\{(r\cos\theta)\,\hat{e}_r + (r\sin\theta)\,\hat{e}_\theta + (r\sin\theta\cos\theta)\varphi\,\hat{e}_\phi\right\}$$

$$=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r\cos\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta r\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(r\sin\theta\cos\varphi)$$

$$=3\cos\theta+2\cos\theta+(-\sin\phi)=5\cos\theta-\sin\phi$$

(b) Find the flux of the above vector field over a *closed* surface that encloses a hemisphere of radius R resting on the xy-plane, with its center at origin and located in the region $z \geq 0$.

Solution: Net flux= $\oint\!\!\oint \vec{A}\bullet\vec{ds} = \iint\limits_{\substack{upper\\hemisphere}} \vec{A}\bullet\vec{ds} + \iint\limits_{\substack{circular\\xy\ plane\\at\ \theta=\frac{\pi}{2}}} \vec{A}\bullet\vec{ds}$



$$\vec{A}=(r\cos\theta)\,\hat{e}_r + (r\sin\theta)\,\hat{e}_\theta + (r\sin\theta\cos\varphi)\hat{e}_\phi$$

$$\vec{A}\bullet\hat{e}_z=(r\cos\theta)\left(\hat{e}_r\bullet\hat{e}_z\right) + (r\sin\theta)\left(\hat{e}_\theta\bullet\hat{e}_z\right)+ (r\sin\theta\cos\varphi)\left(\hat{e}_\phi\bullet\hat{e}_z\right)$$

$$\hat{e}_r=\sin\theta\cos\varphi\,\hat{e}_x+\sin\theta\sin\varphi\,\hat{e}_y+\cos\theta\hat{e}_z$$

$$\hat{e}_\theta=\cos\theta\cos\varphi\,\hat{e}_x+\cos\theta\sin\varphi\,\hat{e}_y-\sin\theta\hat{e}_z$$

$$\hat{e}_\varphi=-\sin\varphi\,\hat{e}_x+\cos\varphi\,\hat{e}_y$$

$$\vec{A}\bullet\hat{e}_z=(r\cos\theta)\left(\cos\theta\right) + (r\sin\theta)\left(-\sin\theta\right)$$

$$= r(\cos^2\theta-\sin^2\theta) = -r \text{ at } \theta=\frac{\pi}{2} \text{ (xy plane)}$$

$$\oint\!\!\oint \vec{A}\bullet\vec{ds} = \iint\limits_{\substack{upper\\hemisphere}} \vec{A}\bullet\vec{ds} + \iint\limits_{\substack{circular\\xy\ plane\\at\ \theta=\frac{\pi}{2}}} \vec{A}\bullet\vec{ds}$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{\varphi=0}^{2\pi}\left(\vec{A}\bullet\hat{e}_r\right)\left(r^2\sin\theta d\theta d\phi\right)+\int_{r=0}^{R}\int_{\varphi=0}^{2\pi}\vec{A}\bullet(rd\varphi dr)(-\hat{e}_z)$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{\varphi=0}^{2\pi}\left(r\cos\theta\right)\left(r^2\sin\theta d\theta d\phi\right)+\int_{r=0}^{R}\int_{\varphi=0}^{2\pi}(r)(rd\varphi dr)$$

$$= \pi R^3 + \frac{2\pi}{3}R^3$$

$$= \frac{5\pi}{3}R^3$$

5 A planet in a remote galaxy rotates rapidly about its own axis. It completes one full rotation in one second. Sketch $T(\lambda)$ *vs* λ for this planet, where $T(\lambda)$ is the time period for the rotation of a Foucault pendulum set in motion on this planet, λ *is the latitude*; $-\frac{\pi}{2} \leq \lambda \leq \frac{\pi}{2}$. **Solution:** $T(\lambda) = \frac{1}{\sin\lambda}$.