

Department of Physics
 Indian Institute of technology Madras
 Select/Special Topics in Classical Mechanics
 Self-Assessment-1 (Questions & Answers)

1. *State whether the following statements are 'TRUE' or 'FALSE' and give reason. The reason should be short, but as rigorous as you can provide.*

- a. The vector $\vec{A} = (\vec{v} \times \vec{H}) - c\hat{e}_\rho$ (wherein c is a constant), defined for a two body mechanical system interacting through a force $cf(|\vec{r}|)\hat{e}_r$, is necessarily a *constant of motion* as long as the force between the two objects is essentially radial (i.e., it is isotropic and has no component along \hat{e}_θ or along \hat{e}_ϕ).

Solution: False

The LRL vector $\vec{A} = (\vec{v} \times \vec{H}) - c\hat{e}_\rho$ is a constant of motion only when the force $f(|\vec{r}|)$ is inversely proportional to r^2 .

- b. The work done $\delta W = q\vec{v} \times \vec{B} \cdot \vec{\delta r}$ by a magnetic field \vec{B} in displacing an electric charge q , which is moving at a velocity \vec{v} , through an infinitesimal displacement $\vec{\delta r}$, is a *pseudo-scalar*.

Solution: False

Lorentz force ($\vec{F} = q\vec{v} \times \vec{B}$) and $\vec{\delta r}$ are both polar vectors. A pseudo scalar includes dot product of a polar vector with an axial vector.

- c. If the Lagrangian of a mechanical system which is free to move in the Cartesian XY plane is independent of the coordinate 'y' but depends on 'x', then $\dot{p}_x = 0$.

Solution: False

Momentum that is canonically conjugate to a cyclic coordinate (y here) is conserved. Therefore in the present case, $\dot{p}_y = 0$.

2. A particle is subjected to a radial force $\vec{F} = f(|\vec{r}|)\hat{e}_r$.

For this force, prove that: $\vec{F} \times \vec{L} = -mf(r) \left[-\vec{r} \dot{r} + r \dot{\vec{r}} \right]$.

Solution:

$$\begin{aligned}\vec{F} &= f(|\vec{r}|)\hat{e}_r = f(r)\frac{\vec{r}}{r} \\ \vec{F} \times \vec{L} &= \frac{mf(r)}{r} \left[\vec{r} \times (\vec{r} \times \dot{\vec{r}}) \right] \\ &= \frac{mf(r)}{r} \left[\vec{r}(\vec{r} \cdot \dot{\vec{r}}) - r^2 \dot{\vec{r}} \right]\end{aligned}$$

The above equation can be further simplified by noting that

$$\begin{aligned}\vec{r} \cdot \dot{\vec{r}} &= \frac{1}{2} \frac{d}{dt} (\vec{r} \cdot \vec{r}) = r \dot{r} \\ \therefore \vec{F} \times \vec{L} &= \frac{mf(r)}{r} \left[(r \dot{r})\vec{r} - r^2 \dot{\vec{r}} \right] \\ \vec{F} \times \vec{L} &= -mf(r) \left[-\vec{r} \dot{r} + r \dot{\vec{r}} \right]\end{aligned}$$

3. If the Lagrangian of a system is independent of one of the generalized coordinate q, prove that the generalized momentum canonically conjugate to this coordinate is conserved.

Solution:

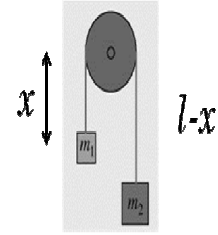
The Euler-Lagrange equation is $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0$.

If the Lagrangian (L) of a system is independent of q, then $\frac{\partial L}{\partial q} = 0$

Then the Euler-Lagrange equation can be written as $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = 0$.

This means that $\frac{\partial L}{\partial \dot{q}} = p$, the generalized momentum is conserved.

4. An Atwood's machine (shown in the figure) has two masses m_1 and m_2 connected by an inextensible string of length l , passing over a frictionless pulley. The potential energy of this system is $V = -m_1gx - m_2g(l-x)$. We consider for the purpose of this problem the pulley's size to be infinitesimal, so no length of the string is unaccounted from the portion that goes over the pulley. Find the generalized momentum conjugate to x , and set up the Euler-Lagrange equation for this system.



Solution:

There is only one independent coordinate x , the position of other weight m_2 is determined by the constraint that the length of the string connecting them is l . The potential energy is

$$V = -m_1gx - m_2g(l-x)$$

The kinetic energy is

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(l-x)^{\cdot 2} = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

The Lagrangian has the form,

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1gx + m_2g(l-x)$$

The generalized momentum, $\frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x}$

$$\frac{\partial L}{\partial x} = (m_1 - m_2)g$$

The Euler-Lagrange equation is $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0$.

Hence, we have

$$(m_1 + m_2)\ddot{x} = (m_1 - m_2)g$$

$$\therefore \ddot{x} = \frac{(m_1 - m_2)g}{(m_1 + m_2)}$$

5. A centrifuge operates at an angular velocity $\vec{\omega} = \omega\hat{e}_z$. Consider the origin of a frame of reference to be at the center of the centrifuge. A particle is located at $\vec{r} = a\hat{e}_\rho + b\hat{e}_z$ in the centrifuge; a and b being positive constants of dimensions [L]. Obtain the magnitudes of the components of the centrifugal force in cylindrical polar coordinate system and fill in the blank spaces in the following expression:

$$\vec{F}_{centrifugal} = \underline{\hspace{2cm}} \hat{e}_\rho + \underline{\hspace{2cm}} \hat{e}_\phi + \underline{\hspace{2cm}} \hat{e}_z.$$

Solution:

The centrifugal force on a particle at $\vec{r} (= a\hat{e}_\rho + b\hat{e}_z)$ is

$$\begin{aligned} \vec{F} &= -m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = -m\omega\hat{e}_z \times \{ \omega\hat{e}_z \times (a\hat{e}_\rho + b\hat{e}_z) \} \\ &= -m\omega\hat{e}_z \times \omega a\hat{e}_\phi \\ &= m\omega^2 a\hat{e}_\rho \end{aligned}$$

$$\vec{F}_{centrifugal} = m\omega^2 a\hat{e}_\rho + 0\hat{e}_\phi + 0\hat{e}_z.$$

