Oscillations and Waves

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Module 1: Oscillations
Lecture 1: Oscillations

Oscillations are ubiquitous. It would be difficult to find something which never exhibits oscillations. Atoms in solids, electromagnetic fields, multi-storeyed buildings and share prices all exhibit oscillations. In this course we shall restrict our attention to only the simplest possible situations, but it should be borne in mind that this elementary analysis provides insights into a diverse variety of apparently complex phenomena.

1.1 Simple Harmonic Oscillators SHO

We consider the spring-mass system shown in Figure 1.1. A massless spring, one of whose ends is fixed has its other attached to a particle of mass m which is free to move. We choose the origin $x = 0$ for the particle's motion at the position where the spring is unstretched. The particle is in stable equilibrium at this position and it will continue to remain there if left at rest. We are interested in a situation where the particle is disturbed from equilibrium. The particle experiences a restoring force from the spring if it is either stretched or compressed. The spring is assumed to be elastic which means that it follows Hooke's law where the force is proportional to the displacement $F = -kx$ with

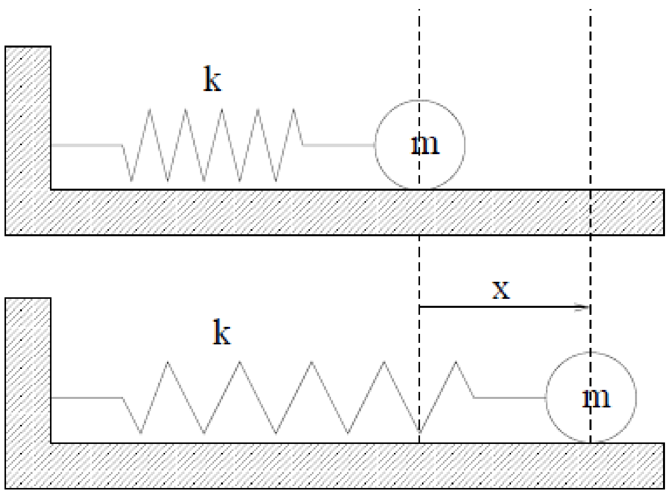


Figure 1.1: Spring-mass system

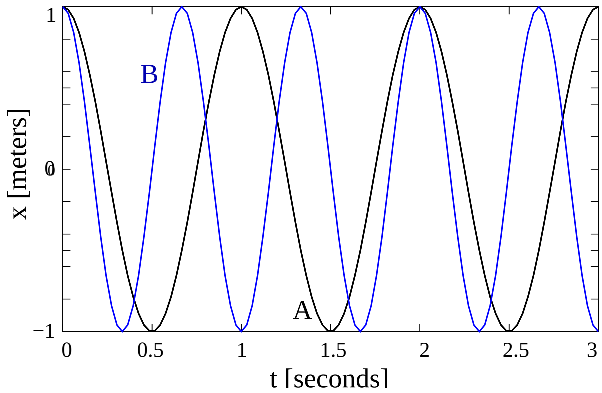


Figure 1.2: Displacement of the oscillator as a function of time for two different frequencies.

spring constant k.

The particle's equation of motion is

$$m\frac{d^2x}{dt^2} = -kx \tag{1.1}$$

which can be written as

$$\ddot{x} + \omega_0^2 x = 0 \tag{1.2}$$

where the dots $\ddot{}$ denote time derivatives and

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{1.3}$$

It is straightforward to check that

$$x(t) = A\cos(\omega_0 t + \phi) \tag{1.4}$$

is a solution to eq. (1.2).

We see that the particle performs sinusoidal oscillations around the equilibrium position when it is disturbed from equilibrium. The angular frequency ω_0 of the oscillation depends on the intrinsic properties of the oscillator. It determines the time period

$$T = \frac{2\pi}{\omega_0} \tag{1.5}$$

and the frequency $\nu = 1/T$ of the oscillation. Figure 1.2 shows oscillations for two different values of ω_0.

Problem 1: What are the values of ω_0 for the oscillations shown in Figure 1.2? What are the corresponding spring constant k values if $m = 1\,\mathrm{kg}$?

Solution: For A $\omega_0 = 2\pi\, s^{-1}$ and $k = (2\pi)^2\,\mathrm{Nm}^{-1}$; For B $\omega_0 = 3\pi\, s^{-1}$ and $k = (3\pi)^2\,\mathrm{Nm}^{-1}$

The amplitude A and phase ϕ are determined by the initial conditions. Two initial conditions are needed to completely specify a solution. This follows

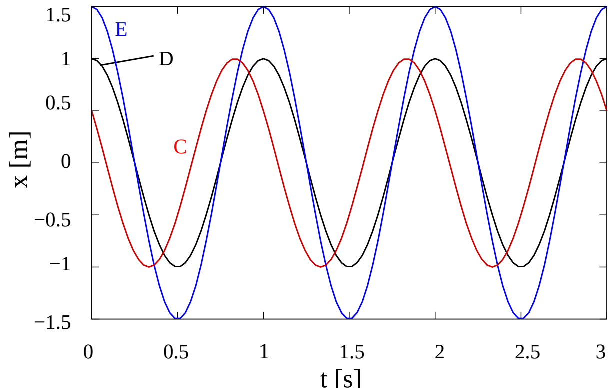


Figure 1.3: Displacement of the oscillator as a function of time for different initial conditions.

from the fact that the governing equation (1.2) is a second order differential equation. The initial conditions can be specified in a variety of ways, fixing the values of $x(t)$ and $\dot{x}(t)$ at $t = 0$ is a possibility. Figure 1.3 shows oscillations with different amplitudes and phases.
Problem 2: What are the amplitude and phase of the oscillations shown in Figure 1.3?
Solution: For C, A=1 and $\phi = \pi/3$; For D, A=1 and $\phi = 0$; For E, A=1.5 and $\phi = 0$;

1.2 Complex Representation.

Complex numbers provide are very useful in representing oscillations. The amplitude and phase of the oscillation can be combined into a single complex number which we shall refer to as the complex amplitude

$$\tilde{A} = Ae^{i\phi} \,. \tag{1.6}$$

Note that we have introduced the symbol ˜ (tilde) to denote complex numbers. The property that

$$e^{i\phi} = \cos\phi + i\sin\phi \tag{1.7}$$

allows us to represent any oscillating quantity $x(t) = A\cos(\omega_0 t + \phi)$ as the real part of the complex number $\tilde{x}(t) = \tilde{A}e^{i\omega_0 t}$,

$$\tilde{x}(t) = Ae^{i(\omega_0 t+\phi)} = A[\cos(\omega_0 t + \phi) + i\sin(\omega_0 t + \phi)] \,. \tag{1.8}$$

We calculate the velocity v in the complex representation $\tilde{v} = \dot{\tilde{x}}$. which gives us

$$\tilde{v}(t) = i\omega_0\tilde{x} = -\omega_0 A[\sin(\omega_0 t + \phi) - i\cos(\omega_0 t + \phi)] \,. \tag{1.9}$$

Taking only the real part we calculate the particle's velocity

$$v(t) = -\omega_0 A\sin(\omega_0 t + \phi) \,. \tag{1.10}$$

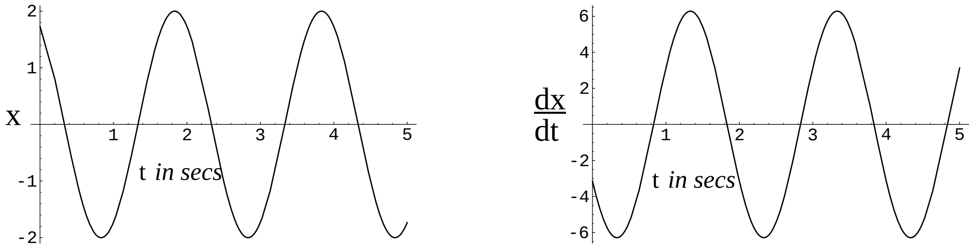


Figure 1.4: Displacement and velocity as a function of time.

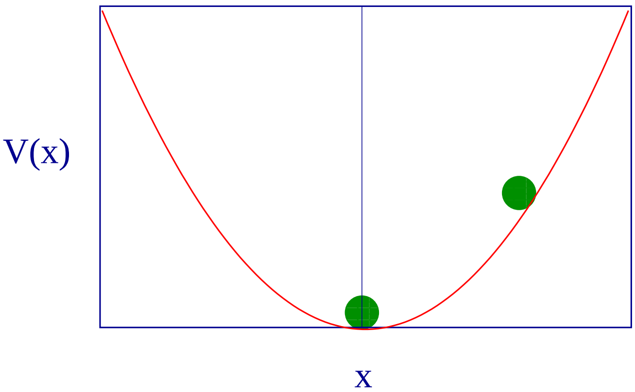


Figure 1.5: Harmonic oscillator potential energy.

The complex representation is a very powerful tool which, as we shall see later, allows us to deal with oscillating quantities in a very elegant fashion.

Figure (1.4) shows the plots of displacement and velocity of the particle described by the equations, (1.8) and (1.10) for amplitude, $A = 2$ units, phase, $\phi = 30°$ and angular frequency, $\omega_0 = \pi$ rad/sec.

Problem 3: A SHO has position x_0 and velocity v_0 at the initial time $t = 0$. Calculate the complex amplitude $\tilde{A}$ in terms of the initial conditions and use this to determine the particle's position $x(t)$ at a later time t.

Solution: The initial conditions tell us that $\mathrm{Re}(\tilde{A}) = x_0$ and $\mathrm{Re}(i\omega_0\tilde{A}) = v_0$. Hence $\tilde{A} = x_0 - iv_0/\omega_0$ which implies that $x(t) = x_0\cos(\omega_0 t) + (v_0/\omega_0)\sin(\omega_0 t)$.

1.3 Energy.

In a spring-mass system the particle has a potential energy $V(x) = kx^2/2$ as shown in Figure 1.5. This energy is stored in the spring when it is either compressed or stretched. The potential energy of the system

$$U = \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi) = \frac{1}{4}m\omega_0^2 A^2\{1 + \cos[2(\omega_0 t + \phi)]\} \tag{1.11}$$

oscillates with angular frequency $2\omega_0$ as the spring is alternately compressed and stretched. The kinetic energy $mv^2/2$

$$T = \frac{1}{2}m\omega_0^2 A^2 \sin^2(\omega_0 t + \phi) = \frac{1}{4}m\omega_0^2 A^2 \{1 - \cos[2(\omega_0 t + \phi)]\} \tag{1.12}$$

shows similar oscillations which are exactly π out of phase.

In a spring-mass system the total energy oscillates between the potential energy of the spring (U) and the kinetic energy of the mass (T). The total energy $E = T + U$ has a value $E = m\omega_0^2 A^2/2$ which remains constant.

The average value of an oscillating quantity is often of interest. We denote the time average of any quantity $Q(t)$ using $\langle Q \rangle$ which is defined as

$$\langle Q \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} Q(t)dt \,. \tag{1.13}$$

The basic idea here is to average over a time interval T which is significantly larger than the oscillation time period.

It is very useful to remember that $\langle \cos(\omega_0 t + \phi) \rangle = 0$. This can be easily verified by noting that the values $\sin(\omega_0 t + \phi)$ are bound between -1 and $+1$. We use this to calculate the average kinetic and potential energies both of which have the same values

$$\langle U \rangle = \langle T \rangle = \frac{1}{4}m\omega_0^2 A^2 \,. \tag{1.14}$$

The average kinetic and potential energies, and the total energy are all very conveniently expressed in the complex representation as

$$E/2 = \langle U \rangle = \langle T \rangle = \frac{1}{4}m\tilde{v}\tilde{v}^* = \frac{1}{4}k\tilde{x}\tilde{x}^* \tag{1.15}$$

where * denotes the conjugate of a complex number.

Problem 4: The mean displacement of a SHO $\langle x \rangle$ is zero. The root mean square (rms.) displacement $\sqrt{\langle x^2 \rangle}$ is useful in quantifying the amplitude of oscillation. Verify that the rms. displacement is $\sqrt{\tilde{x}\tilde{x}^*/2}$.

Solution: $\sqrt{\langle x^2(t) \rangle} = \sqrt{A^2 \langle \cos^2(\omega_0 t + \phi) \rangle} = \sqrt{A^2/2} = \sqrt{\tilde{A}e^{i\omega t}\tilde{A}^* e^{-i\omega t}/2} = \sqrt{\tilde{x}\tilde{x}^*/2}$

1.4 Why study the SHO?

What happens to a system when it is disturbed from stable equilibrium? This question that arises in a large variety of situations. For example, the atoms in many solids (eg. NACl, diamond and steel) are arranged in a periodic crystal as shown in Figure 1.6. The periodic crystal is known to be an equilibrium configuration of the atoms. The atoms are continuously disturbed from their

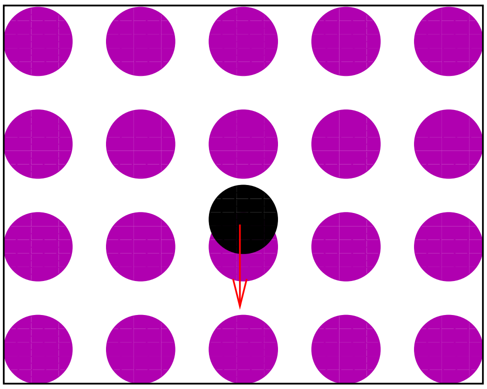


Figure 1.6: Atoms in a Crystal.

equilibrium positions (shown in Figure 1.6) as a consequence of random thermal motions and external forces which may happen to act on the solid. The study of oscillations in the atoms disturbed from their equilibrium position is very interesting. In fact the oscillations of the different atoms are coupled, and this gives rise to collective vibrations of the whole crystal which can explain properties like the specific heat capacity of the solid. We shall come back to this later, right now the crucial point is that each atom behaves like a SHO if we assume that all the other atoms remain fixed. This is generic to all systems which are slightly disturbed from stable equilibrium.

We now show that any potential $V(x)$ is well represented by a SHO potential in the neighbourhood of points of stable equilibrium. The origin of x is chosen so that the point of stable equilibrium is located at $x = 0$. For small values of x it is possible to approximate the function $V(x)$ using a Taylor series

$$V(x) \approx V(x)_{x=0} + \left(\frac{dV(x)}{dx}\right)_{x=0} x + \frac{1}{2}\left(\frac{d^2V(x)}{dx^2}\right)_{x=0} x^2 + ... \tag{1.16}$$

where the higher powers of x are assumed to be negligibly small. We know that at the points of stable equilibrium the force vanishes *ie.* $F = -dV(x)/dx = 0$ and $V(x)$ has a minimum.

$$k = \left(\frac{d^2V(x)}{dx^2}\right)_{x=0} > 0\,. \tag{1.17}$$

This tells us that the potential is approximately

$$V(x) \approx V(x)_{x=0} + \frac{1}{2}kx^2 \tag{1.18}$$

which is a SHO potential. Figure 1.7 shows two different potentials which are well approximated by the same SHO potential in the neighbourhood of the point of stable equilibrium. The oscillation frequency is exactly the same for particles slightly disturbed from equilibrium in these three different potentials.

The study of SHO is important because it occurs in a large variety of situations where the system is slightly disturbed from equilibrium. We discuss a few simple situations.

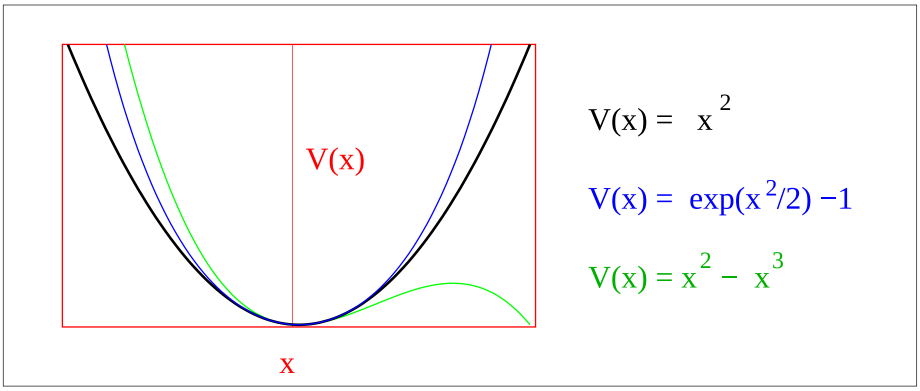


Figure 1.7: Various Potentials.

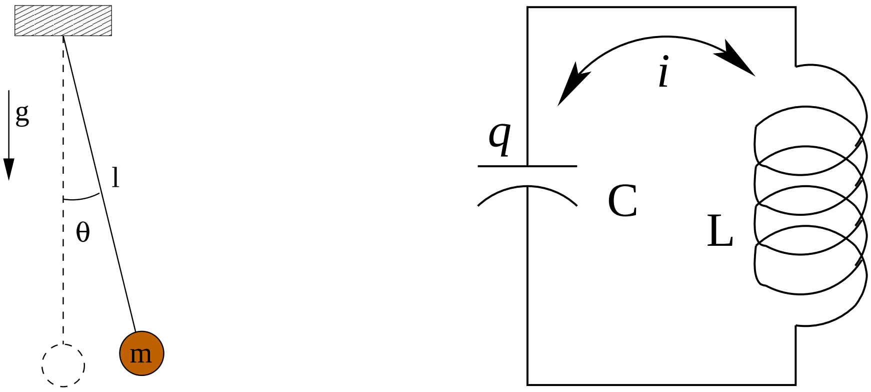


Figure 1.8: (a) Simple Pendulum and (b) LC Circuit.

Simple pendulum

The simple pendulum shown in Figure 1.8(a) is possibly familiar to all of us. A mass m is suspended by a rigid rod of length l, the rod is assumed to be massless. The gravitations potential energy of the mass is

$$V(\theta) = mgl[1 - \cos\theta] . \tag{1.19}$$

For small θ we may approximate $\cos\theta \approx 1 - \theta^2/2$ whereby the potential is

$$V(\theta) = \frac{1}{2} mgl\theta^2 \tag{1.20}$$

which is the SHO potential. Here $dV(\theta)/d\theta$ gives the torque not the force. The pendulum's equation of motion is

$$I\ddot{\theta} = -mgl\theta \tag{1.21}$$

where $I = ml^2$ is the moment of inertia. This can be written as

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{1.22}$$

which allows us to determine the angular frequency

$$\omega_0 = \sqrt{\frac{g}{l}} \tag{1.23}$$

LC Oscillator

The LC circuit shown in Figure 1.8(b) is an example of an electrical circuit which is a SHO. It is governed by the equation

$$L\dot{I} + \frac{Q}{C} = 0 \tag{1.24}$$

where L refers to the inductance, C capacitance, I current and Q charge. This can be written as

$$\ddot{Q} + \frac{1}{LC}Q = 0 \tag{1.25}$$

which allows us to identify

$$\omega_0 = \sqrt{\frac{1}{LC}} \tag{1.26}$$

as the angular frequency.

Torsional pendulum

The equation for the torsional pendulum (figure 1.9(a)) is the following.

$$I\frac{d^2\theta}{dt^2} + \kappa\theta = 0, \tag{1.27}$$

where I is the moment of inertia of the object undergoing torsional oscillation about the axis of rotation and κ is the torsional constant. Angular frequency can be read off directly as $\omega_0 = \sqrt{\frac{\kappa}{I}}$ and hence the time period, $T = 2\pi\sqrt{\frac{I}{\kappa}}$.

Physical pendulum or Compound pendulum

The equation of motion for a compound pendulum shown in (Figure 1.9(b)) is,

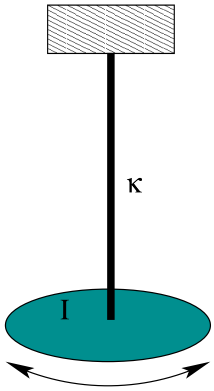
$$I\frac{d^2\theta}{dt^2} = -Mgd\sin\theta, \tag{1.28}$$

where I is the moment of inertia about an axis perpendicular to the plane of oscillations through the point of suspension. For small oscillations($\theta < 4^\circ$) one can write the above equation (1.29) approximately as,

$$\frac{d^2\theta}{dt^2} + \frac{Mgd}{I}\theta = 0. \tag{1.29}$$

The above gives time period as $T = 2\pi\sqrt{\frac{I}{Mgd}}$.

Problem 5: Obtain the simple pendulum results as a special case of the compound pendulum.



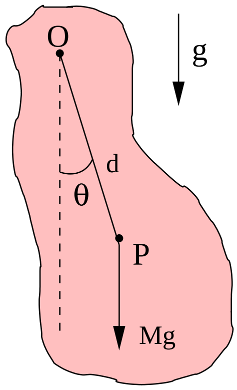


Figure 1.9: (a) Torsional pendulum and (b) Physical pendulum.

Problems

6. An empty tin can floating vertically in water is disturbed so that it executes vertical oscillations. The can weighs 100 gm, and its height and base diameter are 20 and 10 cm respectively. [a.] Determine the period of the oscillations. [b.] How much mercury need one pour into the can to make the time period 1s? (0.227 Seconds, 1.73 cm)

7. A SHO with $\omega_0 = 2\,\mathrm{s}^{-1}$ has initial displacement and velocity 0.1 m and $2.0\,\mathrm{ms}^{-1}$ respectively. [a.] At what distance from the equilibrium position does it come to rest? [b.] What are the rms. displacement and rms. velocity? What is the displacement at $t = \pi/4\,\mathrm{s}$?

8. A SHO with $\omega_0 = 3\,\mathrm{s}^{-1}$ has initial displacement and velocity 0.2 m and $2\,\mathrm{ms}^{-1}$ respectively. [a.] Expressing this as $\tilde{x}(t) = \tilde{A}e^{i\omega_0 t}$, determine $\tilde{A} = a + ib$ from the initial conditions. [b.] Using $\tilde{A} = Ae^{i\phi}$, what are the amplitude A and phase ϕ for this oscillator? [c.] What are the initial position and velocity if the phase is increased by $\pi/3$?

9. A particle of mass $m = 0.3\,\mathrm{kg}$ in the potential $V(x) = 2e^{x^2/L^2}\,\mathrm{J}$ ($L = 0.1\,\mathrm{m}$) is found to behave like a SHO for small displacements from equilibrium. Determine the period of this SHO.

10. Calculate the time average $\langle x^4 \rangle$ for the SHO $x = A\cos\omega t$.