Exam for "Ideas and Methods in Condensed Matter Theory"

- Consider a system of bosons with average density $\bar{n} = 1/2$ on the triangular lattice, which has a low-temperature state with "$\sqrt{3} \times \sqrt{3}$" density-wave order that spontaneously breaks lattice-translation and rotation symmetries by rearranging the boson density to take on mean-values $(\bar{n}+m, \bar{n}-m, \bar{n})$ on the three sublattices of the triangular lattice. In other words, the mean-density on one of the three sublattices, say the a sublattice is $n_a = 0.5 + m$, on another, say the b sublattice is $n_b = 0.5 - m$ and on the third is $n_c = 0.5$.

  Here m is a fraction less than half in magnitude, and there are clearly six such symmetry breaking states with equal free energies. The system spontanously freezes into one of them at low enough temperature. Assuming that the boson number at a site can only take on two values (0 and 1), the system also has "particle-hole" symmetry, under which we can interchange occupied and empty sites—since this transforms the average density $\bar{n}$ to $1 - \bar{n}$, it is a symmetry of the system when $\bar{n} = 1/2$.

  a) Consider the order parameter

  $$\psi \quad = \quad n_a + n_b e^{2\pi i/3} + n_c e^{4\pi i/3} \; . \tag{1}$$

  What are the transformation properties of ψ under lattice translations and rotations of the triangular lattice?

  b) What is the transformation of ψ under particle-hole transformation?

  c) Write down the power-series expansion for the Landau potential of the system as a function of ψ to sixth-order in ψ, using arbitrary coefficients for each distinct term in the expansion. Remember ψ is a complex number and it might help you to first write all terms that do not depend on the phase of ψ and then write down terms that do depend on the phase of ψ. Also remember that every term you write down must be invariant under all transformations considered in a) and b) above.

  d) The sign of one of the coefficients is fixed by the nature of the ordered low-temperature density-wave order. Which coefficient and how?

[Hint: Work out the phase of ψ corresponding to the observed pattern of densities at low temperature. Ask which term controls the phase of ψ.]

e) This potential is supplemented by the "kinetic-energy" of fluctuations of the local order parameter in space. To capture this, we promote ψ to a field that depends on the spatial coordinate $\vec{x}$ and add terms involving gradients of ψ. The simplest and most important of these is a term

$$(\vec{\nabla}\psi) \cdot (\vec{\nabla}\psi^*) \tag{2}$$

f) If fluctuations in the magnitude of ψ are unimportant, argue that the resulting theory is identical to an xy model with a six-fold anisotropy.

g) As discussed in the SAQ for Module 6, a standard duality transformation of the xy model interchanges the roles of vortices with vorticity m and *m-fold anisotropies*, *i.e.* terms of the form

$$-\epsilon_m \int d^2r \cos(m\theta(\vec{r})) \,, \tag{3}$$

and sends the stiffness g to g^{-1} under this duality transformation. This transformation also interchanges the roles of the vortex fugacity $\tilde{\epsilon}_m$ for m-fold vortices and ϵ_m the coefficient of the m-fold anisotropy term.

Using this for the xy model with 6-fold anisotropy, one can straightforwardly obtain the RG flow equations to leading order in both ϵ_6 and the vortex fugacity $\tilde{\epsilon}$ assuming we have a fugacity only for single-vortices:

$$\begin{aligned}
\frac{dg}{dl} &= \frac{36\pi^2}{2}\epsilon_6^2 \exp(-36\pi/2g) - \frac{\pi^2\tilde{\epsilon}^2}{2} g^2 \exp(-\pi g/2) \\
\frac{d\epsilon_6}{dl} &= (2 - 36/2g)\epsilon_p \\
\frac{d\tilde{\epsilon}}{dl} &= (2 - g/2)\tilde{\epsilon} \,,
\end{aligned} \tag{4}$$

Use the above equations to demonstrate that the low-temperature density wave ordered state melts on increasing temperature via two phase transitions, with an intermediate temperature regime in which ψ has power-law correlations. What are the allowed values of the power-law exponent in this intermediate-temperature phase?

[Hint: Identify a stable fixed-line in the above equations which has $\tilde{\epsilon} = 0$ and $\epsilon_6 = 0$]