

Self Assessment - Module 5

- Consider two species of bosonic atoms trapped in a cold-atom experiment and living in an optical lattice potential that forms a two dimensional square lattice. Label the two species by $\alpha = 1, 2$. The Hamiltonian reads

$$H = -t \sum_{\langle ij \rangle, \alpha} (b_{i\alpha}^\dagger b_{j\alpha} + b_{j\alpha}^\dagger b_{i\alpha}) + U \sum_i (1 - \sum_\alpha n_{i\alpha})^2 \quad (1)$$

where $n_{i\alpha} = b_{i\alpha}^\dagger b_{i\alpha}$, for each i and α , is the number of particles at site i of type α , i denotes sites of a square lattice, and $\langle ij \rangle$ denotes nearest neighbour links of a two dimensional square lattice.

The label α can be thought of as a “spin” label by making the correspondence of $\alpha = 1$ with \uparrow and $\alpha = 2$ with \downarrow , and represents two degenerate hyperfine states of some bosonic atom. This Hamiltonian tells us that the atoms hop from site to neighbouring site without changing their spin, and that there is repulsion between atoms if they occupy the same site [the chemical potential has been absorbed into this repulsion term to model the fact that there is one particle per site on average].

A) Sketch the spectrum of H at $t/U = 0$. In particular show that the lowest energy level has degeneracy $2^{N_{\text{sites}}}$ when N_{sites} is the number of sites in the lattice.

B) Working to second order in degenerate perturbation theory in t/U , derive the effective Hamiltonian that governs the splitting of this degeneracy to this order in perturbation theory.

C) Challenge: Can you repeat this for the next unperturbed energy level?

- In class, we derived by elementary means the effective Hamiltonian governing the low-energy physics of the half-filled one-band Hubbard model in the limit in which the Hubbard U is much bigger than the electronic hopping t . This involved keeping track of “fermionic” minus signs “by hand”.

A) Now, repeat this calculation using the language of second quantization, *i.e.* by writing the Hubbard model in terms of creation and annihilation operators, and using standard formulae of degenerate second-order perturbation theory.

B) Compare the steps of this calculation with the corresponding steps in the previous Bosonic problem. Where does the difference between fermions and bosons play a crucial role?

C) Can you rationalize in simple physical terms the difference between the two answers for the two effective Hamiltonians that govern the $U \gg t$ physics at half-filling in the fermionic and bosonic Hubbard models?