# Self Assessment - Module 4

- Consider a system of fermionic atoms F that can be of two types which we label as $\uparrow$ and $\downarrow$ (you may think of these as two possible polarizations of an internal 'spin' degree of freedom, although in real life examples it is a bit more complicated).

  An $\uparrow$ fermion $F_\uparrow$ can form a bosonic 'molecule' (bound state) B by combining with a $\downarrow$ fermion $F_\downarrow$. Conversely, a B molecule can dissociate into a $F_\uparrow$ and $F_\downarrow$. That is, we have

  $$F_\uparrow + F_\downarrow \rightleftharpoons B$$
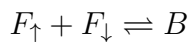
  This 'reaction' (in both directions) is assumed to proceed slowly enough that we may think of the B molecules as being a stable bosonic species in its own right in addition to viewing the $F_\uparrow$ and $F_\downarrow$ as being stable fermionic atoms. However, this reaction does happen quickly enough (in both directions) to ensure that the fermionic and bosonic systems reach a common thermal equilibrium characterized by a single temperature T and a single chemical potential μ in the grand-canonical description (as the number of fermionic atoms N_F is not separately conserved, nor is the number of bosonic molecules N_B).

  The single particle eigenstates available to the fermions are thus labeled by a momentum $\mathbf{p}$ and a 'spin' $\sigma = \pm 1/2$, where $\sigma = +1/2$ corresponds to $F_\uparrow$ and $\sigma = -1/2$ corresponds to $F_\downarrow$. The energy $\epsilon_\sigma^F(\mathbf{p}) = p^2/2m$ independently of σ, while the single particle eigenstates available to the molecules have energy dispersion $\epsilon^B(\mathbf{p}) = -E_B + p^2/4m$, where $E_B \geq 0$ is the binding energy and the $4m$ in the denominator is because the mass of the bosonic molecule is twice the fermionic atom mass m.

  To analyze the equilibrium state of this mixture, we further assume that there are no interparticle interactions we need to include in our description and proceed as follows

  1. Write down an expression in terms of $n_\sigma^F(\mathbf{p})$ and $n^B(\mathbf{p})$ for the conserved total number N_{tot} to which the chemical potential μ couples in our grand canonical description. Here $n_\sigma^F(\mathbf{p})$ and $n^B(\mathbf{p})$ are respectively the number of F particles in state $|\mathbf{p}\sigma\rangle$ and B particles in state $|\mathbf{p}\rangle$, and $\mathbf{p}$ takes on values $2\pi\hbar\vec{n}/L$, with $\vec{n} \equiv$

(n_x, n_y, n_z) integers ranging from $-\infty$ to $+\infty$ as is appropriate for a system confined with periodic boundary conditions to a box of volume L^3.

2. Write down an expression for the total energy in terms of $n_\sigma^F(\mathbf{p})$, $n^B(\mathbf{p})$ and $\epsilon_\sigma^F(\mathbf{p})$ and $\epsilon^B(\mathbf{p})$.

3. Using the above, calculate $Z_{\text{grand}} = Tr \exp(-\beta(H_{\text{tot}} - \mu N_{\text{tot}}))$ to obtain an answer in the form of an infinite product.

4. From this, calculate the expectation value of N_{tot} by differentiating $\log(Z_{\text{grand}})$ with respect to $\beta\mu$.

5. From this, take the naive thermodynamic limit and express the total density $\rho_{\text{tot}} = N_{\text{tot}}/L^3$ as a sum of two integrals.

$$\rho_{\text{tot}} = I_F + I_B$$

where I_F is the contribution of the fermions and I_B the contribution of the bosons.

6. We now need to use this expression to solve for μ in terms of ρ_{tot} and T. To do this, first argue that since the total density is finite, $\mu \leq -E_B$.

For the rest of the problem, consider the simpler case of a marginally bound molecule with $E_B = 0$.

1. Now, argue that for low enough temperature $I_F + I_B$ is bounded above by a small number that will be less than any non-zero ρ_{tot} for the allowed range of μ, thus leading to an inconsistency.

2. Thus, argue that at low enough temperature, we need to carefully include a condensate density contribution $2n_0^B$ in addition to the two integrals, and thereby modify the naive thermodynamic limit expression for ρ_{tot}. Thus

$$\rho_{\text{tot}} = I_F + I_B + 2n_0^B$$

3. Calculate in the $T \to 0$ limit, the value of μ

4. Calculate in the same limit the value of the condensate density n_0^B.

5. Obtain the expectation value of the number of fermionic atoms in the same limit

6. Obtain the expectation value of the total number of bosonic molecules (including condensate) in the same limit

7. Show that the value of μ remains unchanged as the system is heated to slightly above absolute zero.

8. Using the above, calculate the temperature dependence of the condensate density as the temperature is raised slightly above zero. Write your answer in the form $A - BT^p$ and identify the constants A and B and the power p.

9. The transition temperature at which the condensate disappears on heating is the temperature at which this formula predicts that the condensate density is zero. Use this to calculate T_c in terms of total density ρ_{tot} and other system parameters.

*Endnote: If you find this interesting look up 'Resonantly paired fermionic superfluids' by V. Gurarie and L. Radzihowsky in Annals of Physics, **322**, 2-119 (2007)*