Self Assessment - Module 2

1. Consider a modern atom-trap apparatus in which alkali atoms are trapped in a harmonic potential well created by laser beams. Assume that the total number of atoms is fixed, equal to N. Assume that this gas of N atoms has reached equilibrium at temperature T. Assume classical statistical mechanics is a valid approximation in the temperature range of interest.

   The energy function of the gas can of course be written as

   $$E(\mathbf{p}_i, \mathbf{q}_i) = \sum_{i=1}^{N} \left(\frac{\mathbf{p}^2}{2m} + \frac{1}{2} m\omega^2 \mathbf{q}^2 \right)$$

   where ω is the frequency associated with this harmonic trap, m is the mass of the atoms and $\mathbf{p}$, $\mathbf{q}$ are 3-vector momenta and coordinates.

   - Using the Gibbs distribution calculate the Helmholtz free energy $F = -T \log Z_{\text{Gibbs}}$
   - Using the earlier result, calculate the mean energy $\langle E \rangle$ as a function of temperature
   - Using the earlier results, calculate the specific heat (per particle)
   - Using the earlier results, calculate the entropy as a function of temperature.
   - If we want to do a series of experiments with increasing N, how should be adjust the trap frequency ω as we increase N so that the free energy per particle F/N or the entropy S/N tend to a finite non-zero limiting value?

2. Consider an ionic insulating solid where each ion has total angular momentum quantum number $J = 1/2$. In a small external field $B\hat{z}$ along the $\hat{z}$ direction, the energy of the system is $E = -B \sum_{i=1}^{N} m_J(i)$ where each $m_J(i)$ can take values $\pm 1/2$.

   - When the energy is fixed at $E = kB - BN/2$ (where $0 \leq k \leq N$ is an integer), how many different states can the system have at this energy?

- From the above, calculate the entropy as a function of energy $S(E)$.
- From this, calculate the temperature T as a function of energy.
- Invert this to work out $E(T)$.
- Using the above expressions for S and E for large N, calculate $\log Z_G$, where Z_G is the partition function in the Gibbs ensemble at temperature T
- Now calculate $\log(Z_G)$ directly from its definition in terms of the Gibbs distribution at temperature T. Does this agree with your earlier result obtained from expression for S and E?