Introduction to Physics of Nanoparticles and Nano structures

Part I: Physics of Nanoparticles **Questions on Module 5**

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1. Set up the Cartesian coordinates for scattering of em radiation from a particle, taking the direction of propagation of the incident light as the z-axis. Choose any point inside the particle as the origin. The *scattering direction* $\hat{e}_r$ and the forward direction $\hat{e}_z$ define the *scattering plane*.
Let
$$\vec{E}_i = (E_{o||}\hat{e}_{||i} + E_{o\perp}\hat{e}_{\perp i})\, e^{i(q\cdot\vec{x}-\omega t)} \equiv E_{||i}\hat{e}_{||i} + E_{\perp i}\hat{e}_{\perp i}\,, \qquad q = \frac{2\pi n_2}{\lambda},$$
where $\hat{e}_{||i}$ is the unit vector in the scattering plane and $\hat{e}_{\perp i}$ is the unit vector perpendicular to the scattering plane, and $\hat{e}_{||i}$ and $\hat{e}_{\perp i}$ form a right handed triad with $\hat{e}_z$.
Show that $\hat{e}_{||i}$ and $\hat{e}_{\perp i}$ are given by
$$\hat{e}_{\perp i} = \hat{e}_x \sin\phi - \hat{e}_y \cos\phi\,, \qquad \hat{e}_{||i} = \hat{e}_x \cos\phi + \hat{e}_y \sin\phi\,,$$
where ϕ is the azimuthal angle of the scattering direction.

2. Let $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_\phi$ be the orthonormal basis vectors associated with the spherical polar coordinate system r, θ and ϕ.

   (a) Show that
   $$\hat{e}_{\perp i} = -\hat{e}_\phi \qquad \text{and} \qquad \hat{e}_{||i} = \hat{e}_r \sin\theta + \hat{e}_\theta \cos\theta\,.$$

   (b) If E_{xi} and E_{yi} are components of the incident wave, then show that,
   $$E_{||i} = E_{xi}\cos\phi + E_{yi}\sin\phi\,, \qquad E_{\perp i} = E_{xi}\sin\phi - E_{yi}\cos\phi\,.$$

   (c) At sufficiently large distance from the origin ($qr >> 1$), in the so called *far field region*, the scattered field is essentially transverse (*i.e.*, $\hat{e}_r \cdot \vec{E}_s \approx 0$), and has the asymptotic form (consult Jackson),
   $$\vec{E}_s \sim \vec{A}\,\frac{e^{iqr}}{-iqr}\,, \qquad qr >> 1$$
   where $\hat{e}_r \cdot \vec{A} = 0$. Show that the scattered field in the far field region can be written as
   $$\vec{E}_s = E_{||s}\hat{e}_{||s} + E_{\perp s}\hat{e}_{\perp s}\,,$$
   where $\hat{e}_{||s} = \hat{e}_\theta$, $\hat{e}_{\perp s} = -\hat{e}_\phi$.

   (d) Show that that $\hat{e}_{||s}$ is parallel and $\hat{e}_{\perp s}$ is perpendicular to the scattering plane and $\hat{e}_{||s}$, $\hat{e}_{\perp s}$ and $\hat{e}_r$ form a right handed triad.

3. Consider two surfaces parallel to the particle surface, one just outside in the surrounding medium (labeled 2), and the other just inside the particle (labeled 1). Then considering the rate at which the em energy is transferred across the surfaces, derive the boundary conditions for the $\vec{E}$ and $\vec{H}$ fields to ensure conservation of energy.

4. Consider the time averaged Poynting vector $<\vec{S}>$ at any point in medium surrounding the particle to show that it may be written as
$$<\vec{S}> = <\vec{S}_i> + <\vec{S}_s> + <\vec{S}_{ext}>\,,$$
where
$$<\vec{S}_i> = \frac{1}{2}\,\mathrm{Re}(\vec{E}_i \times \vec{H}_i^*), \quad <\vec{S}_s> = \frac{1}{2}\,\mathrm{Re}(\vec{E}_s \times \vec{H}_s^*), \quad <\vec{S}_{ext}> = \frac{1}{2}\,\mathrm{Re}(\vec{E}_i \times \vec{H}_s^* + \vec{E}_s \times \vec{H}_i^*).$$

5. The relation between incident and scattered fields is usually written in matrix form as

$$\begin{pmatrix} E_{||s} \\ E_{\perp s} \end{pmatrix} = \frac{e^{iq(r-z)}}{-iqr} \begin{pmatrix} S_2 & S_3 \\ S_4 & S_1 \end{pmatrix} \begin{pmatrix} E_{||i} \\ E_{\perp i} \end{pmatrix},$$

where the elements S_1, S_2, S_3 and S_4 of the so called "amplitude scattering matrix" depend in general on the scattering angle θ, the azimuthal angle ϕ, the shape and the material properties of the particle.
On the other hand, the relevant Stokes parameters of the waves scattered by the particle are

$$I_s =< E_{||s}E^*_{||s} + E_{\perp s}E^*_{\perp s} >, \qquad Q_s =< E_{||s}E^*_{||s} - E_{\perp s}E^*_{\perp s} >,$$
$$U_s =< E_{||s}E^*_{\perp s} + E_{\perp s}E^*_{||s} >, \qquad V_s = i < E_{||s}E^*_{\perp s} - E_{\perp s}E^*_{||s} >,$$

where a multiplying factor $q/(2\omega\mu)$ is suppressed for brevity.
The relationship between the incident and scattered Stokes parameters follow from the "Scattering matrix", or Mueller matrix given by

$$\begin{pmatrix} I_s \\ Q_s \\ U_s \\ V_s \end{pmatrix} = \frac{1}{q^2r^2} \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{pmatrix} \begin{pmatrix} I_i \\ Q_i \\ U_i \\ V_i \end{pmatrix}.$$

Show that

$$S_{11} = \frac{1}{2}(|S_1|^2 + |S_2|^2 + |S_3|^2 + |S_4|^2), \quad S_{12} = \frac{1}{2}(|S_2|^2 - |S_1|^2 + |S_4|^2 - |S_3|^2),$$
$$S_{13} = \mathrm{Re}(S_2S_3^* + S_1S_4^*), \qquad S_{14} = \mathrm{Im}(S_2S_3^* - S_1S_4^*),$$
$$S_{21} = \frac{1}{2}(|S_2|^2 - |S_1|^2 - |S_4|^2 + |S_3|^2), \quad S_{12} = \frac{1}{2}(|S_2|^2 + |S_1|^2 - |S_4|^2 - |S_3|^2),$$
$$S_{23} = \mathrm{Re}(S_2S_3^* - S_1S_4^*), \qquad S_{24} = \mathrm{Im}(S_2S_3^* + S_1S_4^*),$$
$$S_{31} = \mathrm{Re}(S_2S_4^* + S_1S_3^*), \qquad S_{32} = \mathrm{Re}(S_2S_4^* - S_1S_3^*),$$
$$S_{33} = \mathrm{Re}(S_1S_2^* + S_3S_4^*), \qquad S_{34} = \mathrm{Im}(S_2S_1^* + S_4S_3^*),$$
$$S_{41} = \mathrm{Im}(S_4S_2^* + S_1S_3^*), \qquad S_{42} = \mathrm{Im}(S_4S_2^* - S_1S_3^*),$$
$$S_{43} = \mathrm{Im}(S_1S_2^* - S_3S_4^*), \qquad S_{44} = \mathrm{Re}(S_1S_2^* - S_3S_4^*),$$

6. show that

   (a) For unpolarized incident beam, the scattered light is partially polarized with the degree of polarization given by
   $$P = \sqrt{(S_{21}^2 + S_{31}^2 + S_{41}^2)/S_{11}^2}.$$

   (b) For the incident light completely polarized parallel to a particular scattering plane, the degree of polarization is
   $$P = \frac{\sqrt{(S_{21}+S_{22})^2 + (S_{31}+S_{32})^2 + (S_{41}+S_{42})^2}}{(S_{11}+S_{12})}.$$

7. Consider the incident radiation to be x-polarized, (*i.e.*, $\vec{E}_i = E\,\hat{e}_x$). Then sufficiently far away from the particle, the far-field expression for the scattered em waves can be used to write

$$E_s \sim \frac{e^{iq(r-z)}}{-iqr}\vec{X}\,E, \qquad H_s \sim \frac{q}{\omega\mu}(\hat{e}_r \times \vec{E}_s),$$

where $\hat{e}_r \cdot \vec{X} = 0$. The vector $\vec{X}$ denotes the vector scattering amplitude for x-polarized light. Show that

$$\vec{X} = (S_2\cos\phi + S_3\sin\phi)\,\hat{e}_{||s} + (S_4\cos\phi + S_1\sin\phi)\,\hat{e}_{\perp s},$$

where S_is are the elements of the amplitude scattering matrix.

8. Show that the "Extinction Cross Section" C_{ext}, is described by

$$C_{ext} = \frac{4\pi}{q^2} \, \mathrm{Re}\left[\left(\vec{X} \cdot \hat{e}_x \right)_{\theta=0} \right] \, ,$$

and the "Scattering cross sections" C_{sca} is described by

$$C_{sca} = \int_{4\pi} \frac{|\vec{X}|^2}{q^2} \, d\Omega \; = \int_0^{\pi} \int_0^{2\pi} \frac{|\vec{X}|^2}{q^2} \sin\theta \, d\theta \, d\phi \, .$$

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