

Homework problems in Electrodynamics

The way to confirm that you have understood something is to see if you are able to calculate. Hence these are questions designed to give the student a lot of practice with analytical as well as numerical calculations. Most of these have already appeared in the lectures at appropriate places. Please note the following:

- The questions are often long, more like homework problems than examination problems, and some of them can be approached in multiple ways. Some of the problems make the student complete parts of the derivations that are not given completely in the lecture notes.
- Solutions to these problems have not been provided. The student is expected to think independently about the problems, and take the help of available experts.
- Some of the questions require numerical calculations using either a programming language like C / Fortran, or a software like Mathematica / Maple / Matlab. If the knowledge of programming is not expected / softwares are not available, then these questions may be skipped.
- Many questions ask for plots to be made, since they can give a clearer intuitive picture. Once in a while, the exact values of quantities to be used for plotting are not given, it is a good skill to be able to choose values of parameters that bring out the important features in the plots. The plots may be made by hand, or by using any available plotting software.
- Some questions also ask for “commenting” on the results, at these points it is a good idea to try to appreciate the physical significance of the results.

3 Module 3

3.1 Motion of charges in EM fields

3.1.1 Particle in uniform electric field

Generalize the analysis of a charge moving in a uniform electric field to the case where the particle is moving at an angle θ with the electric field at $t = 0$.

3.1.2 Parallel and constant $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$

Let $\vec{\mathbf{E}} = E_z \hat{\mathbf{z}}$ and $\vec{\mathbf{B}} = B_z \hat{\mathbf{z}}$. Initially, the particle has $\vec{\mathbf{v}} = (v_{x0}, 0, v_{z0})$.

- Show that the solutions for the coordinates $x(t), y(t), z(t)$ are of the form

$$x = \frac{p_T}{eB_z} \sin \phi, \quad y = \frac{p_T}{eB_z} \cos \phi, \quad z = \frac{\mathcal{E}_0}{eE_z} \cosh \left(\frac{E_z \phi}{cB} \right).$$

Determine p_T in terms of the initial conditions given above.

- Draw the trajectory, and comment on the differences between the relativistic and non-relativistic case.

3.1.3 Orthogonal and constant $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$

Take $\vec{\mathbf{E}} = E_x \hat{\mathbf{x}}$ and $\vec{\mathbf{B}} = B_z \hat{\mathbf{z}}$. Take the initial velocity of the particle to be $\vec{\mathbf{v}} = (v_{x0}, v_{y0}, v_{z0})$.

- Find the trajectory of the particle.
- This problem involves some rather complicated algebra and different initial conditions may give rise to qualitatively different trajectories. It is advised to solve this problem numerically on a computer for different sets of initial conditions (even if you get an analytical answer) and comment on the results.

3.2 Retarded potentials

3.2.1 The geometrical factor in Lienard-Wiechert potential

Show that $f(t') = t' - t + \frac{r(t')}{c}$ leads to

$$\frac{\partial f}{\partial t'} = 1 - \frac{\vec{v}(t') \cdot \hat{\mathbf{r}}(t')}{c}$$

where $\hat{\mathbf{r}}(t')$ is a unit vector along the direction of $\vec{\mathbf{r}}(t')$.

3.2.2 Solving for t_r

Show that, given $\vec{\mathbf{x}}, t$ and $\vec{\mathbf{x}}_0(t')$, the implicit equation for t_r can have at most one solution. What happens when there is no solution ?

3.2.3 Orthogonally moving charges

Two charges q_1 and q_2 are moving with uniform velocities along the x and y axis respectively. Their position vectors are given as

$$\vec{\mathbf{x}}_1 = v_1 t \hat{\mathbf{x}}, \quad \vec{\mathbf{x}}_2 = v_2 t \hat{\mathbf{y}}.$$

- Calculate the potentials $\phi(x, y, z, t)$ and $\vec{\mathbf{A}}(x, y, z, t)$ due to the charge q_1 .
- Hence calculate the fields $\vec{\mathbf{E}}(x, y, z, t)$ and $\vec{\mathbf{B}}(x, y, z, t)$ due to the charge q_1 .
- Draw a diagram showing the positions of q_1 and q_2 at an arbitrary time t . Qualitatively show the directions of $\vec{\mathbf{E}}(\vec{\mathbf{x}}_2, t)$ and $\vec{\mathbf{B}}(\vec{\mathbf{x}}_2, t)$. Point out the important features.
- Calculate the force $\vec{\mathbf{F}}_{12}$ on the charge q_2 due to the charge q_1 .
- Calculate, and show in the figure, the force $\vec{\mathbf{F}}_{21}$ on the charge q_1 due to the charge q_2 . Comment on the relative directions of $\vec{\mathbf{F}}_{12}$ and $\vec{\mathbf{F}}_{21}$.

Your answers should be in terms of x, y, z, t, v_1, v_2 and other universal constants, but no other variables. There is no need to combine terms to simplify them.

3.3 Radiation from charges in linear motion

3.3.1 Cherenkov cone and energy of relativistic particles

In a water Cherenkov neutrino detector, a high energy muon neutrino interacts with a nucleus, producing a highly relativistic muon that gives out Cherenkov radiation (refractive index of water: $4/3$). This radiation is detected by a photosensitive plane (made of photomultiplier tubes).

Two muons (mass = 100 MeV) of energies 1 GeV and 10 GeV, respectively, are produced at the same point and travel in a direction normal to the photosensitive plane at a distance 40 m from the point of production. Sketch the pattern of light seen on the photosensitive plane, for both the muons (in the same figure). Show all the relevant distances.

[Assume that the muons, once produced, are absorbed in water within a relatively short distance (as compared to 40 m).]

3.3.2 Calculating $\vec{\mathbf{B}}(\vec{\mathbf{x}}, t)$ for an accelerating charge

Starting with

$$\frac{4\pi\epsilon c^2}{q} \vec{\mathbf{B}}(\vec{\mathbf{x}}, t) = \nabla \times \left(\frac{\vec{\mathbf{v}}(t_r)}{s(t_r)} \right) \Big|_t,$$

- Calculate $\vec{\mathbf{B}}$ and separate it into a component independent of $\vec{\mathbf{a}}$ and a component linear in $\vec{\mathbf{a}}$.
- Show that, the non-radiative part is

$$\vec{\mathbf{B}}(\vec{\mathbf{x}}, t)_{\vec{\mathbf{a}}=0} = \frac{q}{4\pi\epsilon_0 c^2} \frac{1}{s^3 \gamma^2} \vec{\mathbf{v}}(t_r) \times \vec{\mathbf{r}}(t_r)$$

- Show that, for the radiative component,

$$\vec{\mathbf{B}}_{\text{rad}}(\vec{\mathbf{x}}, t) = \frac{\vec{\mathbf{r}}(t_r) \times \vec{\mathbf{E}}(\vec{\mathbf{x}}, t)}{r(t_r)c}$$

- From the expressions for $\vec{\mathbf{E}}(\vec{\mathbf{x}}, t)$ and $\vec{\mathbf{B}}(\vec{\mathbf{x}}, t)$, calculate the Poynting vector. Hence argue that the accelerating charge indeed radiates nonzero energy to infinity.

3.3.3 Similarity with electric dipole radiation

Show that the expressions for radiation fields for a charge accelerating along a straight line with a small velocity are equivalent to an electric dipole. Find the relation between the electric dipole \vec{p} and the acceleration \vec{a} of the charge.

3.3.4 Radiated energy for constant deceleration

For a charge accelerating and undergoing a linear motion,

- Show that the magnetic field \vec{B} reduces to the form $\vec{B} = \vec{r} \times \vec{E}/(rc)$ where $\vec{r} = \vec{x}(t) - \vec{x}'(t_r)$.
- Calculate the total energy radiated in Bremsstrahlung when the speed of a charge q decreases at a constant rate a , from v_0 to 0.

3.3.5 Particle losing energy at a constant rate

A relativistic particle is losing energy at a constant rate $\mathcal{R} = dE/dt'$ while moving through a material in a straight line. In the process, the speed of the particle decreases from $v = 0.9c$ to $v = 0$.

- Plot the power radiated as a function of $\cos \theta$ (the angle between \vec{v} and \vec{r}), when the speed of the particle is $v = 0.9c$, $v = 0.5c$ and $v = 0.1c$ (on the same plot, showing the relative magnitudes, in appropriate units).
- Calculate the total energy radiated by the particle in the form of Bremsstrahlung radiation. You may need to integrate numerically.

3.4 Radiation from charge in circular motion

3.4.1 Dependence of radiated power on φ and θ

For a charge moving in a circle with speed v (see the notation in the lectures)

- Plot the φ -dependence of dU/dt in the x - y plane, for $v = 0.5c$, $v = 0.9c$ and $v = 0.99c$ on the same plot in appropriate units. (You may have to use a logarithmic scale.) Comment on the this angular dependence.

- Plot the average power radiated by the charge as a function of θ for $v = 0.5c$, $v = 0.9c$ and $v = 0.99c$ on the same plot in appropriate units. (You may have to use a logarithmic scale.) Comment on the this angular dependence.

3.4.2 Dependence of Synchrotron radiation on the boost

Plot dU/dt_r as a function of $\tilde{\theta}$, for two values of $\tilde{\phi} : 0, \pi/2$ and three values of $\gamma : 1, 10, 100$. Comment on your results.

3.4.3 Angular dependence and total radiated power

For a charge moving in a circle with speed v (see the notation in the lectures)

- Show the angular distribution of the radiated power as a SphericalPlot3D (Mathematica notation) for $\gamma = 1, 10, 100$ in appropriate units. Comment on the angular dependence.
- Numerically perform the angular integration of dU/dt_r (without the approximation of small $\tilde{\theta}$) to calculate the total power radiated for $\gamma = 1, 10, 100$ in appropriate units. Compare the answers with those obtained analytically by using $\cos \tilde{\theta} \approx 1 - \tilde{\theta}^2$.

3.4.4 Synchrotron radiation for X-ray production

Electrons are to be accelerated in a circle of 10 cm radius in order to generate X-rays of energy $\sim 1keV$.

- Estimate the speed the electrons need to be accelerated to, and hence the frequency ω_0 .
- What is the kinetic energy of the electrons ? hence, how much magnetic field will be needed to keep the electrons in this orbit ?

3.4.5 Radiation from circular motion in a magnetic field

Let a charge q be moving in the xy plane with a constant speed v , under the influence of an external constant magnetic field $\vec{B} = B_0\hat{z}$. The instantaneous velocity of the charge is \vec{v} .

- By explicitly calculating the RHS of the equation of motion

$$mc(du^i/ds) = qF^{ik}u_k ,$$

determine the components of the 4-acceleration \mathbf{a}^i in terms of the components of $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$. Hence write down the 3-acceleration $\vec{\mathbf{a}}$.

- Neglecting the loss of energy due to radiation, the charge will keep on moving in a circle. Determine the radius of the circle and the frequency of revolution of the charge, in terms of $\vec{\mathbf{v}}$, $\vec{\mathbf{B}}$, and energy \mathcal{E} .
- The accelerating charge gives rise to the radiation fields

$$\vec{\mathbf{E}}_{\text{rad}} = \frac{q}{4\pi\epsilon_0 c^2 s^3} \vec{\mathbf{r}} \times (\vec{\mathbf{r}}_{\vec{\mathbf{v}}} \times \vec{\mathbf{a}}) , \quad \vec{\mathbf{B}}_{\text{rad}} = \frac{1}{rc} (\vec{\mathbf{r}} \times \vec{\mathbf{E}}_{\text{rad}}) .$$

When $v \ll c$, calculate the magnitude and direction of the Poynting vector $\vec{\mathbf{N}}(\vec{\mathbf{r}})$ of the radiation, in terms of $\vec{\mathbf{v}}$, $\vec{\mathbf{B}}$ and \mathcal{E} .

- The charge experiences a radiation reaction force, which is given to leading order in γ as

$$\mathbf{f}^i = \frac{q^4}{6\pi\epsilon_0 m^2 c^5} (F_{kl} u^l) (F^{km} u_m) u^i .$$

Calculate the rate of energy loss of the charge in terms of $\vec{\mathbf{v}}$, $\vec{\mathbf{B}}$ and \mathcal{E} .

3.4.6 Power loss at particle accelerators

The power radiated from an accelerated electron may be written as

$$P = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{e^2}{c^3} |\vec{\mathbf{a}}|^2$$

where $\vec{\mathbf{a}}$ is the relativistic acceleration of the charge, defined as $\vec{\mathbf{a}} = (1/m)d\vec{\mathbf{p}}/d\tau$. Here τ is the proper time (in the frame of the electron).

At the particle colliders, electrons are accelerated to a relativistic speed $v \approx c$ and kept circulating in a ring with radius R .

- Calculate $|\vec{\mathbf{a}}|$ in terms of R and $\gamma = E/(mc^2)$ for the electron. Neglect the change in γ due to the energy loss.

- At the LEP accelerator at CERN, electrons of $E=100$ GeV were kept circulating in a ring of $R=4$ km. Calculate the energy lost by an electron (in GeV) due to Synchrotron radiation while completing one circle in the ring.
- If protons are accelerated instead of electrons to the same energy, by what factor will the Synchrotron loss increase / decrease ?

3.5 Radiation reaction force

3.5.1 Energy loss for ultra-relativistic charges

The radiation reaction force on an ultra-relativistic charge is $\vec{f}_{\text{rad}} \propto \gamma^2 \propto \mathcal{E}^2$, where \mathcal{E} is the kinematic energy of the charge.

Starting from $d\mathcal{E}/dx = -k(x)\mathcal{E}^2$, determine how the kinematic energy of the charge will change as a function of time / distance travelled.

3.5.2 Validity of radiation reaction force

Find the conditions of validity of the radiation reaction force in the ultra-relativistic limit.

3.5.3 Small- v limit of the radiation reaction force

Can one obtain the small- v limit of the radiation reaction force starting from the relativistic expression ?

3.6 Passage of radiation through matter

3.6.1 Dilute electron gas

Using the expression for the refractive index of a dilute electron gas with number density N , plot the phase velocity and group velocity of an EM wave as a function of ω . Neglect the damping term and choose an appropriate value for N . Where is the “dilute” nature of the gas relevant ?

3.6.2 Dilute gas as a high-pass filter

Show that a dilute gas of free electrons will not allow an EM wave to propagate through it, if the frequency of the wave is less than a cutoff frequency

ω_{cutoff} . Determine the cutoff frequency in terms of the number density of electrons and other universal constants. For $\omega > \omega_{\text{cutoff}}$, qualitatively plot the behaviour of the wavenumber k as a function of ω .