Homework problems in Electrodymanics

The way to confirm that you have understood something is to see if you are able to calculate. Hence these are questions designed to give the student a lot of practice with analytical as well as numerical calculations. Most of these have already appeared in the lectures at appropriate places. Please note the following:

- The questions are often long, more like homework problems than examination problems, and some of them can be approached in multiple ways. Some of the problems make the student complete parts of the derivations that are not given completely in the lecture notes.

- Solutions to these problems have not been provided. The student is expected to think independently about the problems, and take the help of available experts.

- Some of the questions require numerical calculations using either a programming language like C / Fortran, or a software like Mathematica / Maple / Matlab. If the knowledge of programming is not expected / softeares are not available, then these questions may be skipped.

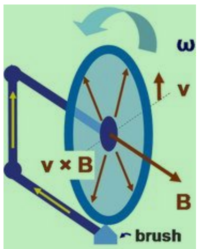
- Many questions ask for plots to be made, since they can give a clearer intuitive picture. Once in a while, the exact values of quantities to be used for plotting are not given, it is a good skill to be able to choose values of parameters that bring out the important features in the plots. The plots may be made by hand, or by using any available plotting software.

- Some questions also ask for "commenting" on the results, at these points it is a good idea to try to appreciate the physical significance of the results.

2 Module 2

2.1 Relativity and Maxwell's equations

2.1.1 Faraday Disc



See the figure. There is a cylindrical bar magnet that is placed along the axis of the disc, so that it produces a uniform magnetic field in a cylindrical zone along the axis of the disc. The conduction loop is completed through the brush, which is made of a conducting material.

Will a current flow if:

- The magnet is stationary and the disc is spinning?
- The disc is stationary and the magnet is spinning about its axis?
- Both the disc and the magnet are spinning with the same angular speed?

2.2 Lorentz transformations of EM fields and waves

2.2.1 Transformation of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields

The transformations for components of ∇' and $\partial/\partial t'$ were obtained in the lectures. Assume the components of $\vec{\mathbf{E}}'$ and $\vec{\mathbf{B}}'$ to be some linear combinations of the components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$, with coefficients that are functions of v. Show that

$$E'_x = E_x \ , \quad E'_y = \gamma(E_y - vB_z) \ , \quad E'_z = \gamma(E_z + vB_y) \ ,$$
$$B'_x = B_x \ , \quad B'_y = \gamma(B_y + \frac{v}{c^2}E_z) \ , \quad B'_z = \gamma(B_z - \frac{v}{c^2}E_y) \ .$$

2.2.2 Intensity of a moving source

A source emitting light of wavelength λ isotropically (Intensity $I(\theta') = I'_0$) is mounted on a rocket moving with a large (relativistic) speed v along x direction.

- Calculate an analytic expression for the intensity $I(\theta) \propto |\vec{\mathbf{E}}|^2$ of the emitted light, as observed in the stationary frame, as a function of θ. (Hint: You may separate $\vec{\mathbf{E}}$ into two components, one in the xy plane, one along the z axis.)

- Plot intensity as a function of θ for $v = 0.5c, v = 0.9c, v = 0.99c$.

2.2.3 Reflection of polarized light

Polarized light ($\vec{\mathbf{E}}$ in the plane of incidence) of frequency ω is incident on an infinitely large dielectric surface (dielectric constant n) at an angle of incidence θ_I. It is partly reflected and partly transmitted.

As observed by an observer moving with a large (relativistic) velocity $\vec{\mathbf{v}}$ towards the dielectric surface, in a direction normal to the surface:

- Determine the angle of incidence θ'_I, the angle of reflection θ'_R, and the angle of transmission θ'_T.

- Calculate the magnitude of the incident $\vec{\mathbf{E}}'$ as observed by this observer in terms of $|\vec{\mathbf{E}}|, \omega, \vec{\mathbf{v}}, \theta_I$.

[Lorentz transformations for $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields:

$$\begin{aligned} \vec{\mathbf{E}}'_\parallel &= \vec{\mathbf{E}}_\parallel \,, \qquad & \vec{\mathbf{E}}'_\perp &= \gamma(\vec{\mathbf{E}}_\perp + \vec{\mathbf{v}} \times \vec{\mathbf{B}}_\perp) \,, \\ \vec{\mathbf{B}}'_\parallel &= \vec{\mathbf{B}}_\parallel \,, \qquad & \vec{\mathbf{B}}'_\perp &= \gamma(\vec{\mathbf{B}}_\perp - \vec{\mathbf{v}} \times \vec{\mathbf{E}}_\perp) \,. \qquad] \end{aligned}$$

2.2.4 Static EM fields due to a charged cylinder

An infinite cylinder of radius R carries a constant current I, and has zero charge density as observed by an observer A. Another observer C travels parallel to the wire with a constant large (relativistic) speed v with respect to A.

- Find $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ observed by C, both inside and outside the cylinder.

- Find the charge density measured by C. Comment on your answer.

2.3 Kinematic quantities in Special Relativity

2.3.1 Acceleration in a moving frame

Calculate the components of acceleration in frame S', in terms of the components of velocity and acceleration in frame S, and the boost.

2.3.2 Force and acceleration

Find the conditions under which $\vec{\mathbf{F}}$ and $\vec{\mathbf{a}}$ can be parallel.

2.3.3 Trajectory, velocity and acceleration

A train is moving with a large (relativistic) speed v in the x direction. A ball is launched from the floor of the carriage at a speed u, making an angle θ' with the horizontal, in the xy plane. Seen from the frame of the train, it goes on a parabolic trajectory and returns to the floor. In the stationary frame, calculate

- the trajectory $(x(t), y(t))$ of the ball
- the velocity $\vec{\mathbf{u}}(t)$
- the acceleration $\vec{\mathbf{a}}(t)$

2.3.4 Velocity and acceleration

A train is moving with a constant large (relativistic) velocity $\vec{\mathbf{v}}$. A person sitting on the train is moving a pendulum in a vertical complete circle of radius R with a constant angular velocity ω. The axis of the circle is horizontal, and normal to the direction of motion of the train.

To a stationary observer outside the train, it appears that the speed of the pendulum is the largest when it is at the bottom of its trajectory. At this point, what does this observer measure as

- the velocity and acceleration of the pendulum bob ?
- the force on the pendulum bob ?

2.4 Relativistic kinematics

2.4.1 Two-body scattering

- Determine all the Lorentz-invariant scalar products involved in the AB→CD scattering in terms of $\mathsf{p}_A \cdot \mathsf{p}_B$ and $\mathsf{p}_A \cdot \mathsf{p}_C$.

- In Compton scattering, if the photon is scattered at an angle θ, what is its frequency after scattering ? Do this problem by the usual conservation of energy and 3-momentum, and also by using the result obtained above.

2.4.2 Particle decays

- If a particle A of mass m_A decays into two particles B and C, of masses m_B and m_C, respectively, calculate the energy of B.

- A moving particle A is observed to decay into three almost massless particles that move in directions orthogonal to each other. If the energies of the decay products are measured to be E_1, E_2, E_3,

  - Determine the mass of the particle A.
  - What was the speed of A ?

  [Keep track of all factors of c.]

2.4.3 Rutherford scattering, invariant q^2 and energy loss

Rutherford scattering describes the process when an electron of mass m_e scatters off a nucleus of a much larger mass m_N. In this process, the energy absorbed by the nucleus is very small, and the electron energy is unchanged, though the direction of the electron changes by an angle θ. The cross section for Rutherford scattering is given by

$$\frac{d\sigma}{d\Omega} = \frac{Z^2 \alpha^2 (\hbar c)^2}{4 |\vec{\mathbf{p}}_e|^2 \beta^2 \sin^4(\theta/2)} \ .$$

where $|\vec{\mathbf{p}}_e| = |\vec{\mathbf{p}}_{e1}| = |\vec{\mathbf{p}}_{e2}|$. Here $\vec{\mathbf{p}}_{e1}$ and $\vec{\mathbf{p}}_{e2}$ are the momenta of the electron before and after the scattering, respectively, Z is the atomic number of the nucleus, α is the fine structure constant, and $\beta = |\vec{\mathbf{v}}|/c$, where $\vec{\mathbf{v}}$ is the velocity of the incoming electron.

Let us describe Rutherford scattering in terms of $q^i = p^i_{e1} - p^i_{e2}$, the 4-momentum transferred by the electron to the nucleus.

- Neglecting the electron mass m_e, calculate the Lorentz invariant quantity q^2 in terms of $|\vec{\mathbf{p}}_e|$ and θ.
- Hence, calculate the cross section $d\sigma/dq^2$.

Now view this process in the reference frame where the electron is originally stationary, and the nucleus scatters on it, transferring an energy $\mathcal{E}$ to it. In this frame,

- Calculate the energy transfer $\mathcal{E}$ in terms of q^2. Do not neglect the electron mass.
- Hence, determine $d\sigma/d\mathcal{E}$.
- Find an upper bound on the energy transfer $\mathcal{E}$.

2.5 Lagrangian formulation

2.5.1 Hamiltonian for a particle in EM field

Using the Hamiltonian

$$H = \sqrt{(\vec{\mathbf{P}} - e\vec{\mathbf{A}})^2 c^2 + m^2 c^4} + q\phi ,$$

determine the equations of motion.

2.5.2 Lorentz force law

Show that the covariant equation of motion

$$mc\frac{d\mathsf{u}_k}{ds} = q(\partial_k \mathsf{A}_m - \partial_m \mathsf{A}_k)u^k = q\,\mathsf{F}_{km}\,\mathsf{u}^m$$

corresponds to the Lorentz force law.

2.5.3 Energy-momentum tensor

For a Lagrangian density $\mathcal{L}(q, \partial_i q)$, the energy-momentum tensor T_i^k is given by

$$T_i^k = \partial_i q(\partial\mathcal{L}/\partial_k q) - \delta_i^k \mathcal{L} \ .$$

For the electromagnetic field in the absence of charges,

$$\mathcal{L} = -(\epsilon_0 c/4) F_{kl} F^{kl} \ .$$

- Taking q as the 4-potential A_m, determine T_i^k in terms of the components of the electromagnetic field tensor F.
- Calculate the components of T_i^k in terms of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$.

2.5.4 Angular momentum tensor in 4-d

An infinitesimal rotation in 4-d is defined as $x'^i - x^i = \delta x^i = x_k \delta\Omega^{ik}$.

- Show that $\delta\Omega_{ik}$ is an antisymmetric tensor.
- For a collection of free particles, the action is $S = -\sum mc \int_a^b ds$. Show that $\delta S = \delta\Omega_{ik} M^{ik}$ where $M^{ik} = (1/2)\sum(p^i x^k - p^k x^i)$. Hence argue that M^{ik} is conserved.
- M^{ik} is the angular momentum 4-tensor. Calculate the components of this tensor in terms of $\vec{\mathbf{r}}, \vec{\mathbf{p}}$ and $\vec{\mathbf{M}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$ of the individual particles.