Homework problems in Electrodymanics

The way to confirm that you have understood something is to see if you are able to calculate. Hence these are questions designed to give the student a lot of practice with analytical as well as numerical calculations. Most of these have already appeared in the lectures at appropriate places. Please note the following:

- The questions are often long, more like homework problems than examination problems, and some of them can be approached in multiple ways. Some of the problems make the student complete parts of the derivations that are not given completely in the lecture notes.

- Solutions to these problems have not been provided. The student is expected to think independently about the problems, and take the help of available experts.

- Some of the questions require numerical calculations using either a programming language like C / Fortran, or a software like Mathematica / Maple / Matlab. If the knowledge of programming is not expected / softeares are not available, then these questions may be skipped.

- Many questions ask for plots to be made, since they can give a clearer intuitive picture. Once in a while, the exact values of quantities to be used for plotting are not given, it is a good skill to be able to choose values of parameters that bring out the important features in the plots. The plots may be made by hand, or by using any available plotting software.

- Some questions also ask for "commenting" on the results, at these points it is a good idea to try to appreciate the physical significance of the results.

1 Module 1

1.1 Uniqueness theorems

1.1.1 Unique vector, given divergence and curl

Show that, given $\nabla \cdot \vec{\mathbf{V}} = s(\vec{\mathbf{x}})$,
and $\nabla \times \vec{\mathbf{V}} = \vec{\mathbf{c}}(\vec{\mathbf{x}})$ (with $\nabla \cdot \vec{\mathbf{c}}(\vec{\mathbf{x}}) = 0$, of course),
if $\vec{\mathbf{V}}$ goes to zero at infinity (fast enough),
then $\vec{\mathbf{V}}$ can be uniquely written in terms of $s(\vec{\mathbf{x}})$ and $\vec{\mathbf{c}}(\vec{\mathbf{x}})$.
Indeed the solution can be given as:

$$V = -\nabla\phi(\vec{\mathbf{x}}) + \nabla \times \vec{\mathbf{A}}(\vec{\mathbf{x}})$$

with

$$\begin{aligned}
\phi(\vec{\mathbf{x}}) &= \frac{1}{4\pi}\int \frac{s(\vec{\mathbf{x}})}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3\vec{\mathbf{x}}' \\
\vec{\mathbf{A}}(\vec{\mathbf{x}}) &= \frac{1}{4\pi}\int \frac{\vec{\mathbf{c}}(\vec{\mathbf{x}})}{|\vec{\mathbf{x}} - \vec{\mathbf{x}}'|} d^3\vec{\mathbf{x}}'
\end{aligned}$$

1.1.2 Unique scalar, given $\nabla^2\phi$ and boundary conditions

Show that, for a scalar $\phi(\vec{\mathbf{x}})$,
given $\nabla^2\phi$ everywhere,
and given $\phi(\vec{\mathbf{x}})$ or $\nabla\phi \cdot \hat{\mathbf{n}}$ on a closed surface,
a unique solution for $\phi(\vec{\mathbf{x}})$ exists.

1.1.3 Unique vector, given $\nabla \times (\nabla \times \vec{\mathbf{A}})$

Show that, for a vector $\vec{\mathbf{A}}(\vec{\mathbf{x}})$,
given $\nabla \times (\nabla \times \vec{\mathbf{A}})$ everywhere,
and given $\vec{\mathbf{A}} \times \hat{\mathbf{n}}$ or $(\nabla \times \vec{\mathbf{A}}) \times \hat{\mathbf{n}}$ on a closed surface,
a unique solution for $\vec{\mathbf{A}}(\vec{\mathbf{x}})$ exists.

1.2 EM waves: reflection and transmission

1.2.1 reflection and transmission coefficients

Let a plane EM wave with frequency ω travelling in free space be incident on the surface of a dielectric with refractive index n and permeability μ_0, at an angle of incidence θ_I.

Let the electric field of the wave be normal to the plane of incidence, i.e. parallel to the surface of the dielectric.

- In terms of the parameter $\alpha = \cos\theta_T / \cos\theta_I$ and $\beta = n$, calculate the reflection coefficient R and the transmission coefficient T.
- Plot these quantities as functions of θ_I. Is there a "Brewster's angle" in this case ?

1.2.2 Polarization from reflection

The polarization of a light beam is defined as

$$P = \frac{I_1 - I_2}{I_1 + I_2}$$

where I_1 and I_2 are the intensities of the two orthogonal polarizations.

If an unpolarized beam of light is incident on a dielectric with refractive index n and permeability μ_0,

- Calculate the polarization of the reflected beam of light as a function of the angle of incidence θ_I.
- Plot this dependence for $n = 1.5$. You may use the results on reflection and transmission coefficients.

1.3 Confined EM waves

1.3.1 Decaying wave at a conducting surface

Inside a conducting medium, an EM wave will propagate as well as decay. Let the form of the plane wave solution be

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 e^{-\kappa x} e^{ikx} e^{-i\omega t} \ .$$

- Find $c\kappa/\omega$ and ck/ω as a functions of $\omega\tau$, where τ is the relaxation time. Show your results in the form of a plot that brings out all the relevant features. Comment on the plot.

- For an EM wave reflecting normally from the surface of a conductor with finite σ, calculate the surface current $\vec{K}$ and the time averaged value of the Poynting vector $\vec{\mathbf{N}}$ into the surface. Hence determine the "surface resistance" R_s.

1.3.2 Coaxial transmission line

Consider a transmission line consisting of two long co-axial cylinders of radii a and b with empty space in between ($a < b$). An AC voltage $V = V_0 e^{-i\omega t}$ is applied between the cables at one end.

- Calculate the electric field $\vec{E}$, the magnetic field $\vec{B}$, and the current I flowing through the transmission line in the form of a TEM wave, as functions of time. What is the average power P transmitted ?

- For $a = 1$ cm and $b = 2$ cm, determine the capacitance, inductance and conductance of the transmission line for the transmission of TEM modes in SI units. (Some of these quantities may have to be defined per unit length, specify these clearly.)

- What are the smallest frequencies at which the TE and TM modes will be transmitted ?

1.3.3 Cylindrical waveguide

Consider the waves propagating in a cylindrical waveguide, with the form

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}(x,y)e^{i(k_z z - \omega t)} \ , \quad \vec{\mathbf{B}} = \vec{\mathbf{B}}(x,y)e^{i(k_z z - \omega t)} \ .$$

- Using Maxwell's equations, calculate E_x, E_y, B_x, B_y in terms of E_z and B_z.

- Determine the second-order differential equations that E_z and B_z should separately satisfy.

1.3.4 Group velocity inside a waveguide

Let the threshold frequency for a rectangular waveguide of size $a \times 2a$ be ω_c.

- Calculate the group velocities v_g as functions of frequency ω (in units of ω_c) for the lowest five TE modes.
- Show their behaviour qualitatively on the same plot, indicating the relevant values of v_g (in units of c) and ω (in units of ω_c) along the axes.

1.3.5 Microwave oven

We want to make a microwave oven that will operate at 10 GHz. The walls of the cavity are to be coated with silver to ensure that not more than 10^{12}th fraction of the energy in EM waves leaks out.

What is the minimum thickness of silver coating needed ? You may make any reasonable assumptions, but state them clearly.

[Useful information:

$$1/(4\pi\epsilon_0) = 9 \times 10^9 N \cdot m^2/C^2 \text{ }, \mu_0/(4\pi) = 10^{-7} N/A^2$$

The resistivity of silver at room temperature is $\sim 15 \; n\Omega \cdot m$.]

1.3.6 TM wave in a rectangular waveguide

For a TM wave propagating through a rectangular waveguide,

$$E_z = A \sin(k_x x) \sin(k_y y) e^{i(k_z z - \omega t)} \;, \quad B_z = 0 \;.$$

Starting from Maxwell's equations for $\nabla \times \vec{\mathbf{E}}$ and $\nabla \times \vec{\mathbf{B}}$, calculate

- the transverse components of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields.
- the Poynting vector (Magnitude and direction)
- the conductance of the waveguide in terms of the parameters of E_z given above.

1.4 EM potentials with sources

1.4.1 Solving the Green's equation

Given that the solutions to the Green's equation

$$\frac{1}{r}\frac{\partial}{\partial r}(rG) + k^2 G = -\delta(r)$$

is of the form

$$G(r) = (A/r)e^{\pm ikr} ,$$

determine the value of A by integrating the equation over a small sphere centered at the origin.

1.4.2 Continuity equation

Using the continuity equation, show that for a monochromatic source,

$$\vec{\mathbf{E}}(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0}\int \left[\frac{[\dot{\rho}]}{cr}\hat{\mathbf{r}} - \frac{[\dot{\vec{\mathbf{J}}}]}{c^2 r}\right] d^3\mathbf{x}'$$

reduces to

$$\vec{\mathbf{E}}(\vec{\mathbf{x}}, t) = \frac{1}{4\pi\epsilon_0 c^2}\int \frac{(\left[\dot{\vec{\mathbf{J}}}(\vec{\mathbf{x}}')\right] \times \vec{\mathbf{r}}) \times \vec{\mathbf{r}}}{r^3} d^3\mathbf{x}'$$

where $r = |\vec{\mathbf{x}} - \vec{\mathbf{x}}'|$.

1.4.3 Retarded potential and Fourier transform

Starting from the "retarded potential" solution for $\vec{\mathbf{A}}(x, t)$, calculate its Fourier transform $\vec{\mathbf{A}}_\omega(\vec{\mathbf{x}})$, and through it, calculate $\vec{\mathbf{B}}_\omega(\vec{\mathbf{x}})$.

1.5 EM radiation

1.5.1 Dipole near an infinite conductor

An infinite grounded conductor in the x-y plane has an oscillating electric dipole $\vec{p} = p_0\hat{z}e^{i\omega t}$ placed at a distance a from it.

- Find the total power radiated.

- Qualitatively sketch the radiation pattern in the x-z plane.
  (If you can give arguments motivating the pattern, there is no need to calculate it explicitly.)

1.5.2 Radiation from two circulating charges

Two charges, $+q$ and $-q$, are kept circulating about their common center of mass O (taken to be the origin), in a circle of radius a, with frequency ω. Calculate

- the electric dipole, magnetic dipole and electric quadrupole components of $\vec{\mathbf{A}}, \vec{\mathbf{B}}, \vec{\mathbf{E}}$ at large distances
- the power radiated per solid angle in these three modes
- the total values of $\vec{\mathbf{A}}, \vec{\mathbf{B}}, \vec{\mathbf{E}}$ at large distances
- the total power radiated per solid angle.
- What fraction of the total power radiated is accounted for by the combined power in ED, MD and EQ modes ? Comment.

1.5.3 Slow turn-on of current

Current is slowly turned on in an infinite straight wire, such that

$$I(t) = \begin{cases} 0 & (t < 0) \\ I_0(t/\tau) & (0 \leq t < \tau) \\ I_0 & (t \geq \tau) \end{cases}$$

Calculate the resulting electric and magnetic fields as functions of time t and perpendicular distance d from the wire. (Note: the exact $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are needed, not just their radiative components.)

1.5.4 Linear antenna

A linear antenna of length L (at $-L/2 < x < L/2$) is fed a current

$$I = I_0 \frac{\sin[k(L/2 - |x|)]}{\sin(kL/2)} \ .$$

Plot the radiation pattern (3-d plot as well as projections in yz and xy planes) for (i) $L = 0.01\lambda$, (ii) $L = 0.5\lambda$, (iii) $L = 1.1\lambda$, (iv) $L = 10\lambda$.