Department Of MECHANICAL ENGINEERING

MODULE 5

Unsteady State Heat Conduction

UNSTEADY HEAT TRANSFER

Many heat transfer problems require the understanding of the complete time history of the temperature variation. For example, in metallurgy, the heat treating process can be controlled to directly affect the characteristics of the processed materials. Annealing (slow cool) can soften metals and improve ductility. On the other hand, quenching (rapid cool) can harden the strain boundary and increase strength. In order to characterize this transient behavior, the full unsteady equation is needed:

$$\rho c \frac{\partial T}{\partial t} = k \nabla^2 T, \text{ or } \frac{1}{\alpha}\frac{\partial T}{\partial t} = \nabla^2 T$$

where $\alpha = \frac{k}{\rho c}$ is the thermal diffusivity

"A heated/cooled body at T_i is suddenly exposed to fluid at T_∞ with a known heat transfer coefficient . Either evaluate the temperature at a given time, or find time for a given temperature."

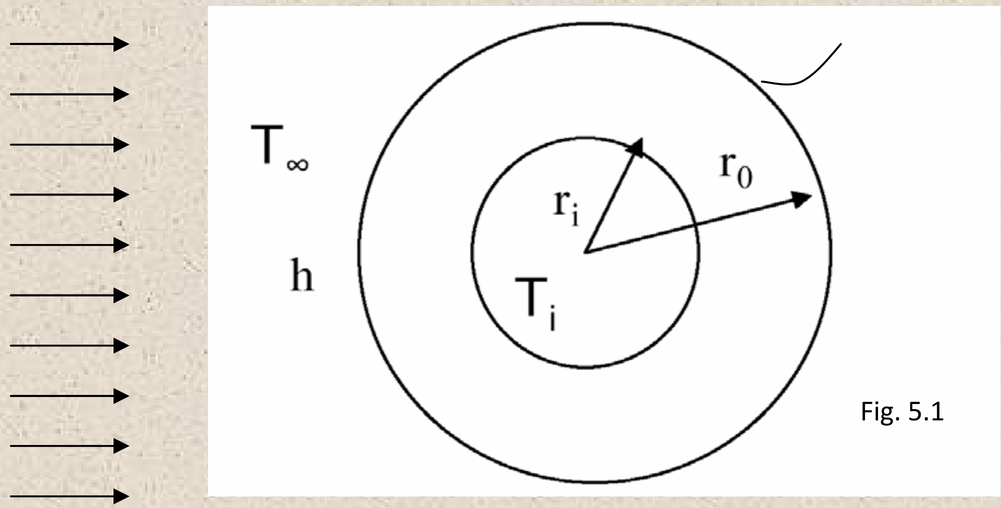


Fig. 5.1

Q: "How good an approximation would it be to say the annular cylinder is more or less *isothermal*?"

A: "Depends on the relative importance of the thermal conductivity in the thermal circuit compared to the convective heat transfer coefficient".

Biot No. Bi

•Defined to describe the relative resistance in a thermal circuit of the convection compared

$$Bi = \frac{hL_c}{k} = \frac{L_c / kA}{1/hA} = \frac{\text{Internal conduction resistance within solid}}{\text{External convection resistance at body surface}}$$

L_c is a characteristic length of the body

Bi→0: No conduction resistance at all. The body is isothermal.

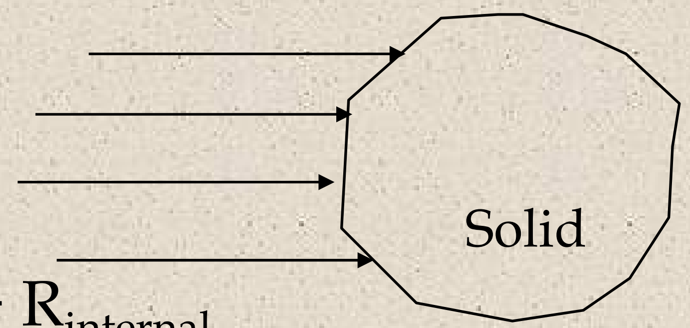
Small Bi: Conduction resistance is less important. The body may still be approximated as isothermal

Lumped capacitance analysis can be performed.

Large Bi: Conduction resistance is significant. The body cannot be treated as isothermal.

Transient heat transfer with no internal resistance: Lumped Parameter Analysis

Valid for Bi<0.1



Total Resistance= $R_{external} + R_{internal}$

GE: $$\frac{dT}{dt} = -\frac{hA}{mc_p}\left(T - T_\infty\right)$$

BC: $$T\left(t = 0\right) = T_i$$

Solution: $let\ \Theta = T - T_\infty$, therefore

$$\frac{d\Theta}{dt} = -\frac{hA}{mc_p}\Theta$$

Lumped Parameter Analysis

$$\Theta_i = T_i - T_\infty$$

$$\ln \frac{\Theta}{\Theta_i} = -\frac{hA}{mc_p} t$$

$$\frac{\Theta}{\Theta_i} = e^{-\frac{hA}{mc_p} t}$$

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-t \Big/ \frac{mc_p}{hA}}$$

- To determine the temperature at a given time, or
- To determine the time required for the temperature to reach a specified value.

Note: Temperature function **only of time** and **not** of space!

Lumped Parameter Analysis

$$\mathrm{T} = \frac{T - T_\infty}{T_0 - T_\infty} = \exp\left(-\frac{hA}{\rho c V}t\right)$$

$$\frac{hA}{\rho c V}t = \left(\frac{hL_c}{k}\right)\left(\frac{k}{\rho c}\right)\frac{1}{L_c}\frac{1}{L_c}t = Bi\,\frac{\alpha}{L_c^{\ 2}}t$$

Thermal diffusivity: $\alpha \equiv \left(\frac{k}{\rho c}\right)$ ($m^2\ s^{-1}$)

Lumped Parameter Analysis

Define Fo as the Fourier number (dimensionless time)

$$Fo \equiv \frac{\alpha}{L_c^{\,2}} t \quad \text{and Biot number} \quad Bi \equiv \frac{hL_C}{k}$$

The temperature variation can be expressed as

$$\mathbf{T} = \exp(-Bi*Fo)$$

where L_c is a characteristic length scale : realte to the size of the solid invloved in the problem

for example , $L_c = \dfrac{r_o}{2}$ (half - radius) when the solid is a cylinder.

$L_c = \dfrac{r_o}{3}$ (one - third radius) when the solid is sphere

$L_c = L$ (half thickness) when the solid is aplane wall with a 2L thickness

Spatial Effects and the Role of Analytic Solutions

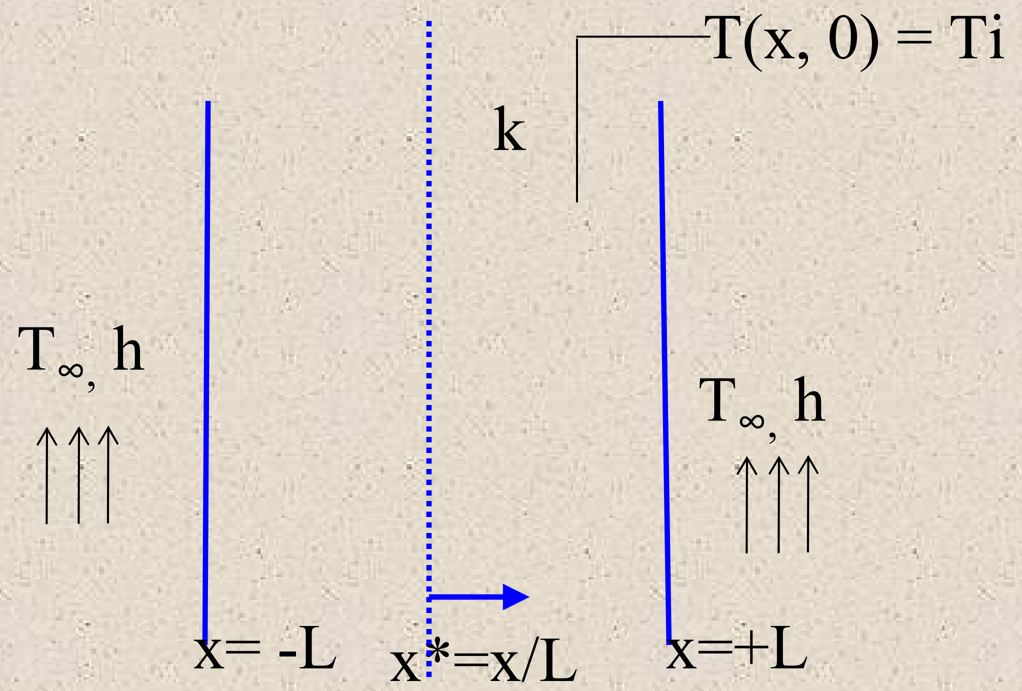
The Plane Wall: Solution to the Heat Equation for a Plane Wall with Symmetrical Convection Conditions

$$\frac{1}{a}\cdot\frac{\partial T}{\partial \tau}=\frac{\partial^2 T}{\partial x^2}$$

$$T(x,0)=T_i$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0}=0$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=l}=h\left[T(L,t)-T_\infty\right]$$



The Plane Wall:

Note: Once spatial variability of temperature is included, there is existence of seven different independent variables.

How may the functional dependence be simplified?

- The answer is **Non-dimensionalisation.** We first need to understand the physics behind the phenomenon, identify parameters governing the process, and group them into meaningful non-dimensional numbers.

Dimensionless temperature difference: $$\theta^* = \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty}$$

Dimensionless coordinate: $$x^* = \frac{x}{L}$$

Dimensionless time: $$t^* = \frac{\alpha t}{L^2} = Fo$$

The Biot Number: $$Bi = \frac{hL}{k_{solid}}$$

The solution for temperature will now be a function of the other non-dimensional quantities

$$\theta^* = f(x^*, Fo, Bi)$$

Exact Solution: $$\theta^* = \sum_{n=1}^{\infty} C_n \exp\left(-\zeta_n^2 Fo\right)\cos\left(\zeta_n x^*\right)$$

$$C_n = \frac{4\sin\zeta_n}{2\zeta_n + \sin(2\zeta_n)} \qquad \zeta_n \tan\zeta_n = Bi$$

The roots (eigenvalues) of the equation can be obtained from tables given in standard textbooks.

The One-Term Approximation $Fo > 0.2$

Variation of mid-plane temperature with time Fo $(x^* = 0)$

$$\theta_0^* = \frac{T - T_\infty}{T_i - T_\infty} \approx C_1 \exp\left(-\zeta_1^2 Fo\right)$$

From tables given in standard textbooks, one can obtain C_1 and ζ_1 as a function of Bi.

Variation of temperature with location (x^*) and time (Fo):

$$\theta^* = \theta_0^* = \cos\left(\zeta_1 x^*\right)$$

Change in thermal energy storage with time:

$$\Delta E_{st} = -Q$$

$$Q = Q_0\left(1 - \frac{\sin\zeta_1}{\zeta_1}\right)\theta_0^*$$

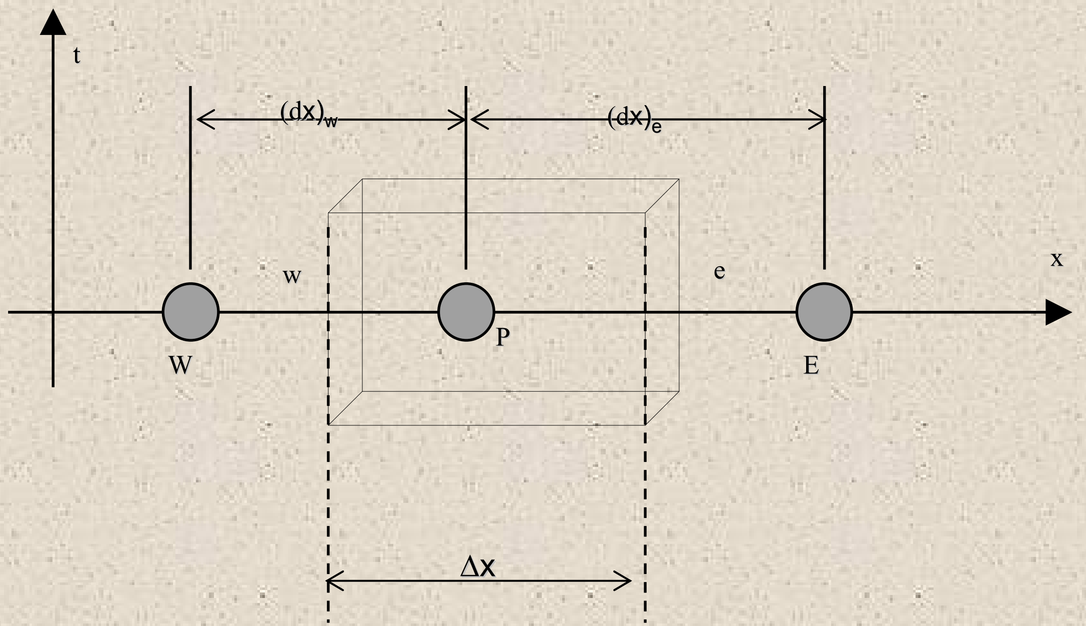
$$Q_0 = \rho c V\left(T_i - T_\infty\right)$$

Numerical Methods for Unsteady Heat Transfer

- Unsteady heat transfer equation, no generation, constant k, one-dimensional in Cartesian coordinate:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x} \right) + S$$

- The term on the left hand side of above eq. is the storage term, arising out of accumulation/depletion of heat in the domain under consideration. Note that the eq. is a partial differential equation as a result of an extra independent variable, time (t). The corresponding grid system is shown in fig. on next slide.



Integration over the control volume and over a time interval gives

$$\int_{t}^{t+\Delta t}\left(\int_{CV}\left(\rho c\frac{\partial T}{\partial t}\right)dV\right)dt = \int_{t}^{t+\Delta t}\left(\int_{cv}\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dV\right)dt + \int_{t}^{t+\Delta t}\left(\int_{CV} S dV\right)dt$$

$$\int_{w}^{e}\left(\int_{t}^{t+\Delta t}\rho c\frac{\partial T}{\partial t}dt\right)dV = \int_{t}^{t+\Delta t}\left(\left(kA\frac{\partial T}{\partial x}\right)_e - \left(kA\frac{\partial T}{\partial x}\right)_w\right)dt + \int_{t}^{t+\Delta t}\left(\bar{S}\Delta V\right)dt$$

If the temperature at a node is assumed to prevail over the whole control volume, applying the central differencing scheme, one obtains:

$$\rho c\left(T^{new}{}_{P} - T_P^{old}\right)\Delta V = \int_{t}^{t+\Delta t}\left(\left(k_e A \frac{T_E - T_P}{\delta x_e}\right) - \left(k_w A \frac{T_P - T_W}{\delta x_w}\right)\right)dt + \int_{t}^{t+\Delta t}\left(\bar{S}\Delta V\right)dt$$

Now, an assumption is made about the variation of T_P, T_E and T_w with time. By generalizing the approach by means of a weighting parameter f between 0 and 1:

$$\int_{t}^{t+\Delta t}\phi_P dt = \phi_P \Delta t = \left[f\phi_P^{new} - (1-f)\phi_P^{old}\right]\Delta t$$

Repeating the same operation for points E and W,

$$\rho c\left(\frac{T_P^{new} - T_P^{old}}{\Delta t}\right)\Delta x = f\left[\left(k_e \frac{T_E^{new} - T_P^{new}}{\delta x_e}\right) - \left(k_w \frac{T_P^{new} - T_W^{new}}{\delta x_w}\right)\right]$$
$$+ (1-f)\left[\left(k_e \frac{T_E^{old} - T_P^{old}}{\delta x_e}\right) - \left(k_w \frac{T_P^{old} - T_W^{old}}{\delta x_w}\right)\right] + \bar{S}\Delta x$$

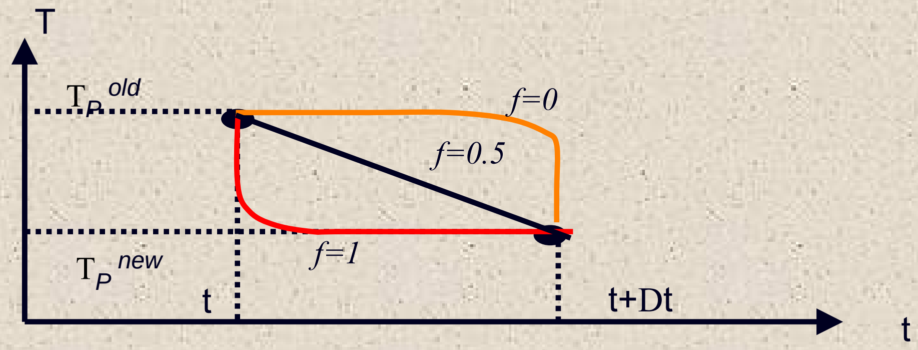
Upon re-arranging, dropping the superscript “new”, and casting the equation into the standard form

$$a_P T_P = a_W \left[f T_W + (1-f) T_W^{old} \right] + a_E \left[f T_E + (1-f) T_E^{old} \right] + \left[a_P^0 - (1-f) a_W - (1-f) a_E \right] T_P^{old} + b$$

$$a_P = \theta \left(a_W + a_E \right) + a_P^0 \qquad a_P^0 = \rho c \frac{\Delta x}{\Delta t} \qquad a_W = \frac{k_w}{\delta x_w} \qquad a_E = \frac{k_e}{\delta x_e} \qquad b = \bar{S} \Delta x$$

The time integration scheme would depend on the choice of the parameter f. When $f = 0$, the resulting scheme is “explicit”; when $0 < f \leq 1$, the resulting scheme is “implicit”; when $f = 1$, the resulting scheme is “fully implicit”, when $f = 1/2$, the resulting scheme is “Crank-Nicolson”.

Variation of T within the time interval Δt for different schemes



Explicit scheme

Linearizing the source term as and setting $f = 0$

$$a_P T_P = a_W T_W^{old} + a_E T_E^{old} + \left[a_P^0 - (a_W + a_E)\right] T_P^{old} + S_u$$

$$a_P = a_P^0 \qquad a_P^0 = \rho c \frac{\Delta x}{\Delta t} \qquad a_W = \frac{k_w}{\delta x_w} \quad a_E = \frac{k_e}{\delta x_e}$$

For stability, all coefficients must be positive in the discretized equation. Hence,

$$a_P^0 - (a_W + a_E - S_P) > 0$$

;

; $$\longrightarrow \quad \rho c\frac{\Delta x}{\Delta t} - \left(\frac{k_w}{\delta x_w} + \frac{k_e}{\delta x_e}\right) > 0 \quad \longrightarrow \quad \rho c\frac{\Delta x}{\Delta t} > \frac{2k}{\Delta x} \quad \longrightarrow \quad \Delta t < \rho c\frac{(\Delta x)^2}{2k}$$

;

; The above limitation on time step suggests that the explicit scheme becomes very expensive to improve spatial accuracy. Hence, this method is generally not recommended for general transient problems.

Crank-Nicolson scheme

Setting $f = 0.5$, the Crank-Nicolson discretisation becomes:

$$a_P T_P = a_E\left(\frac{T_E + T_E^{old}}{2}\right) + a_W\left(\frac{T_W + T_W^{old}}{2}\right) + \left[a_P^0 - \frac{a_E}{2} - \frac{a_W}{2}\right]T_P^0 + b$$

$$a_P = \frac{1}{2}(a_E + a_W) + a_P^0 - \frac{1}{2}S_P \qquad a_P^0 = \rho c\frac{\Delta x}{\Delta t} \qquad a_W = \frac{k_w}{\delta x_w} \qquad a_E = \frac{k_e}{\delta x_e} \qquad b = S_u + \frac{1}{2}S_p T_p^{old}$$

;

; For stability, all coefficient must be positive in the discretized
; equation, requiring

$$a_P^0 > \frac{a_E + a_W}{2} \longrightarrow \Delta t < \rho c \frac{(\Delta x)^2}{k}$$

The Crank-Nicolson scheme only slightly less restrictive than the explicit method. It is based on central differencing and hence it is second-order accurate in time.

The fully implicit scheme

Setting $f = 1$, the fully implicit discretisation becomes:

$$a_P T_P = a_E T_E + a_W T_W + a_P^0 T_P^{old}$$

$$a_P = a_P^0 + a_E + a_W - S_P \qquad a_P^0 = \rho c \frac{\Delta x}{\Delta t} \qquad a_W = \frac{k_w}{\delta x_w} \qquad a_E = \frac{k_e}{\delta x_e}$$

General remarks:

A system of algebraic equations must be solved at each time level. The accuracy of the scheme is first-order in time. The time marching procedure starts with a given initial field of the scalar ϕ^0. The system is solved after selecting time step Δt. For the implicit scheme, all coefficients are positive, which makes it unconditionally stable for any size of time step. Hence, the implicit method is recommended for general purpose transient calculations because of its robustness and unconditional stability.