MODULE 4

Multi-Dimensional Steady State Heat Conduction

MULTIDIMENSIONAL HEAT TRANSFER

Heat Diffusion Equation

$$\rho c_p \frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \dot{q} = k\nabla^2 T + \dot{q}$$

- This equation governs the Cartesian, temperature distribution for a three-dimensional unsteady, heat transfer problem involving heat generation.

- For steady state $\partial / \partial t = 0$

- No generation $\dot{q} = 0$

- To solve for the full equation, it requires a total of six boundary conditions: two for each direction. Only one initial condition is needed to account for the transient behavior.

Two-Dimensional, Steady State Case

For a 2 - D, steady state situation, the heat equation is simplified to

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0,$$ it needs two boundary conditions in each direction.

There are three approaches to solve this equation:

- Numerical Method: Finite difference or finite element schemes, usually will be solved using computers.
- Graphical Method: Limited use. However, the conduction shape factor concept derived under this concept can be useful for specific configurations. (see Table 4.1 for selected configurations)
- Analytical Method: The mathematical equation can be solved using techniques like the method of separation of variables. (refer to handout)

Conduction Shape Factor

- This approach applied to 2-D conduction involving two isothermal surfaces, with all other surfaces being adiabatic. The heat transfer from one surface (at a temperature T_1) to the other surface (at T_2) can be expressed as: $q=Sk(T_1-T_2)$ where k is the thermal conductivity of the solid and S is the conduction shape factor.
- The shape factor can be related to the thermal resistance: $q=Sk(T_1-T_2)=(T_1-T_2)/(1/kS)= (T_1-T_2)/R_t$

  where $R_t = 1/(kS)$
- 1-D heat transfer can use shape factor also. Ex: Heat transfer inside a plane wall of thickness L is $q=kA(\Delta T/L)$, $S=A/L$
- Common shape factors for selected configurations can be found in most textbooks, as also illustrated in Table 4.1.

Table 4.1

Conduction shape factors for selected two-dimensional systems [$q = Sk(T_1 - T_2)$]

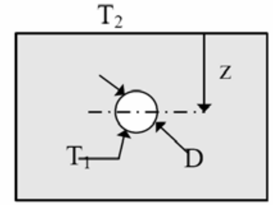
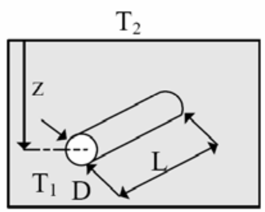
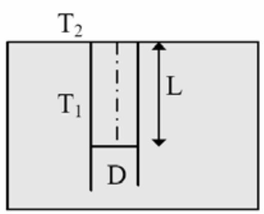
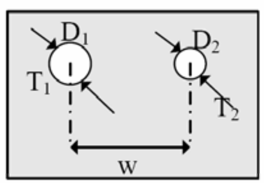
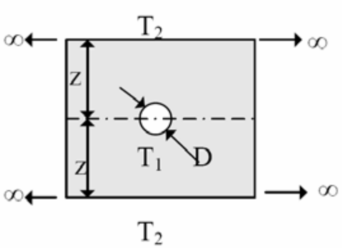
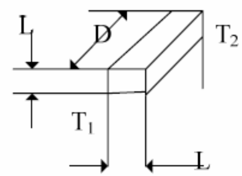
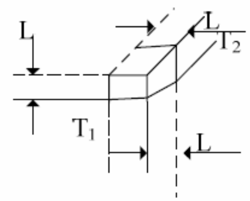
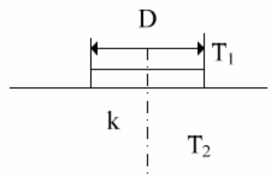
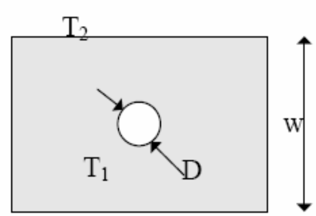
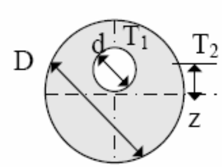
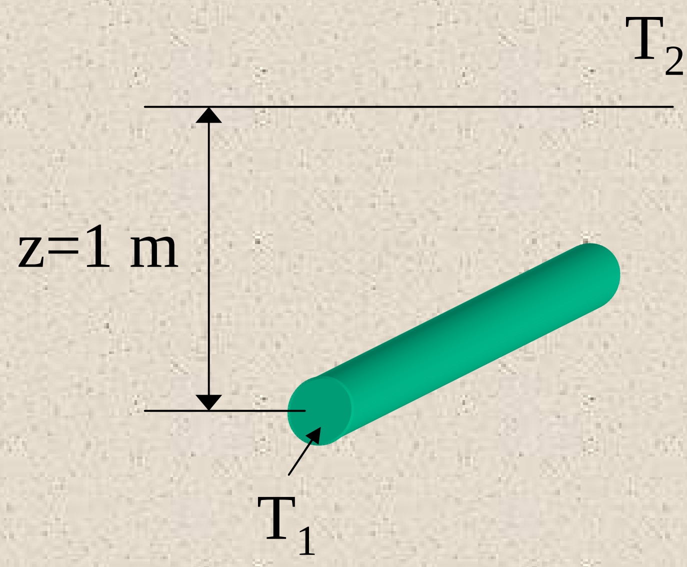
System	Schematic	Restrictions	Shape Factor
Isothermal sphere buried in as finite medium		$z>D/2$	$\frac{2\pi D}{1-D/4z}$
Horizontal isothermal cylinder of length L buried in a semi finite medium		$L>>D$ $L>>D$ $z>3D/2$	$\frac{2\pi L}{\cosh^{-1}(2z/D)}$ $\frac{2\pi L}{\ln(4z/D)}$
Vertical cylinder in a semi finite medium		$L>>D$	$\frac{2\pi L}{\ln(4L/D)}$
Conduction between two cylinders of length L in infinite medium		$L>>D_1,D_2$ $L>>w$	$\frac{2\pi L}{\cosh^{-1}\left(\frac{4w^2-D_1^2-D_2^2}{2D_1D_2}\right)}$
Horizontal circular cylinder of length L midway between parallel planes of equal length and infinite width		$z>>D/2$ $L>>2$	$\frac{2\pi L}{\ln(8z/\pi D)}$

Table 4.1 Continued

System	Schematic	Restrictions	Shape Factor
Conduction through the edge of adjoining walls		D>L/5	0.54D
Conduction through corners of three walls with a temperature difference of ΔT_{1-2} across the walls		L<<length and width of wall	0.15L
Disk of diameter D and T1 on a semi finite medium of thermal conductivity k and T_2		None	2D
Circular cylinder of length L centered in a square solid of equal length		W>D L>>w	$\frac{2\pi L}{\ln(1.08w/D)}$
Eccentric circular cylinder of length L in a cylinder of equal length		D>d L>>D	$\frac{2\pi L}{\cosh^{-1}\left(\frac{D^2+d^2-4z^2}{2Dd}\right)}$

Example

<u>Example:</u> A 10 cm OD uninsulated pipe carries steam from the power plant across campus. Find the heat loss if the pipe is buried 1 m in the ground is the ground surface temperature is 50 ºC. Assume a thermal conductivity of the sandy soil as k = 0.52 w/m K.



The shape factor for long cylinders is found in Table 4.1 as Case 2, with $L >> D$:

$S = 2 \cdot \pi \cdot L / \ln(4 \cdot z/D)$

Where z = depth at which pipe is buried.

$S = 2 \cdot \pi \cdot 1 \cdot m / \ln(40) = 1.7\ m$

Then
$q' = (1.7 \cdot m)(0.52\ W/m \cdot K)(100\ {}^{o}C - 50\ {}^{o}C)$

$q' = 44.2\ W$

Numerical Methods

- Due to the increasing complexities encountered in the development of modern technology, analytical solutions usually are not available. For these problems, numerical solutions obtained using high-speed computer are very useful, especially when the geometry of the object of interest is irregular, or the boundary conditions are nonlinear. In numerical analysis, three different approaches are commonly used: the *finite difference*, the *finite volume* and the *finite element* methods. In heat transfer problems, the finite difference and finite volume methods are used more often. Because of its simplicity in implementation, the finite difference method will be discussed here in more detail.

Numerical Methods (contd…)

- The finite difference method involves:

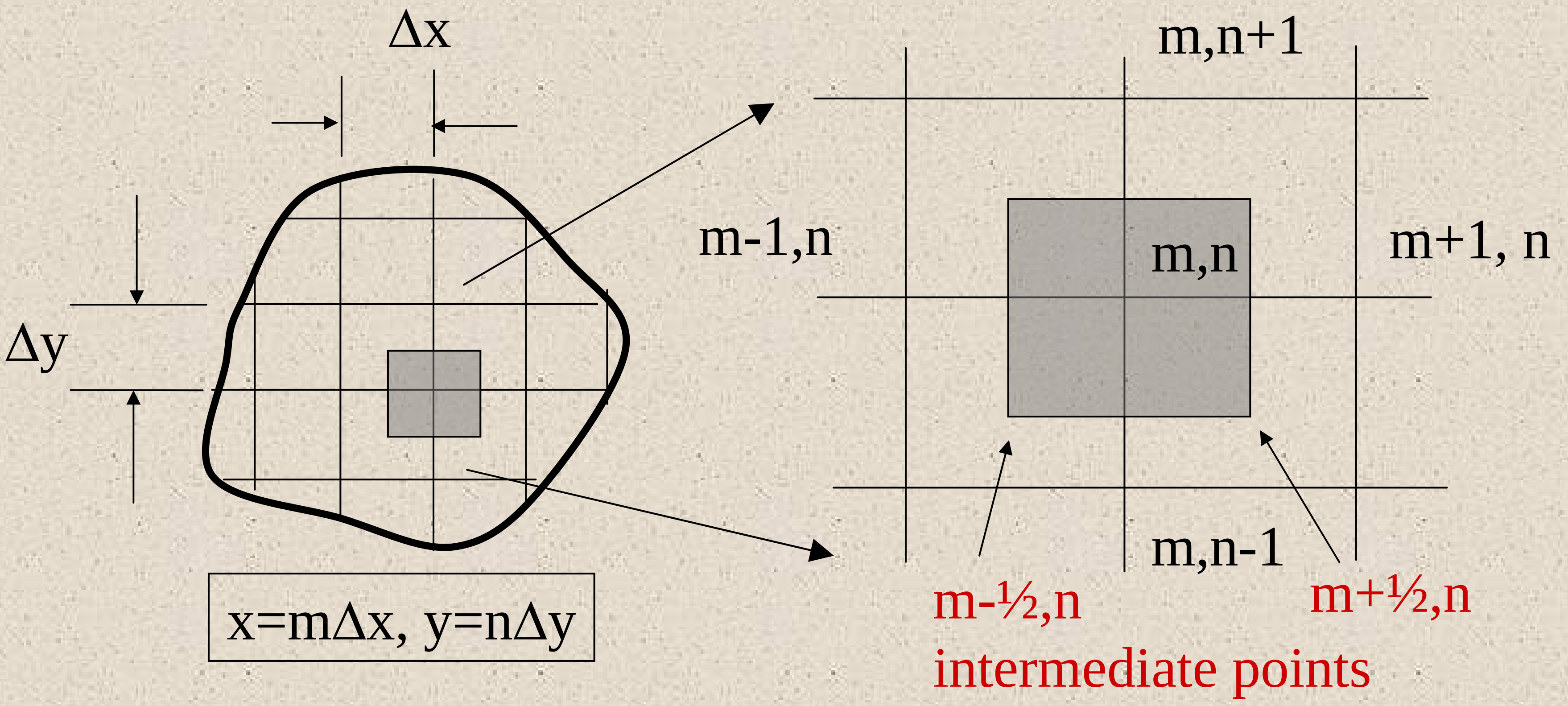
- Establish nodal networks
- Derive finite difference approximations for the governing equation at both interior and exterior nodal points
- Develop a system of simultaneous algebraic nodal equations
- Solve the system of equations using numerical schemes

The Nodal Networks

- The basic idea is to subdivide the area of interest into sub-volumes with the distance between adjacent nodes by Δx and Δy as shown. If the distance between points is small enough, the differential equation can be approximated locally by a set of finite difference equations. Each node now represents a small region where the nodal temperature is a measure of the average temperature of the region.

The Nodal Networks (contd…)

Example



Finite Difference Approximation

Heat Diffusion Equation: $\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$,

where $\alpha = \frac{\mathrm{k}}{\rho \mathrm{C_P} V}$ is the thermal diffusivity

No generation and steady state: $\dot{q}=0$ and $\frac{\partial}{\partial \mathrm{t}} = 0, \Rightarrow \nabla^2 T = 0$

First, approximated the first order differentiation

at intermediate points (m+1/2,n) & (m-1/2,n)

$$\left.\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\right|_{(m+1/2,n)} \approx \left.\frac{\Delta T}{\Delta x}\right|_{(m+1/2,n)} = \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\left.\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\right|_{(m-1/2,n)} \approx \left.\frac{\Delta T}{\Delta x}\right|_{(m-1/2,n)} = \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

Finite Difference Approximation (contd...)

Next, approximate the second order differentiation at m,n

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{\left.\partial T/\partial x\right|_{m+1/2,n} - \left.\partial T/\partial x\right|_{m-1/2,n}}{\Delta x}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2}$$

Similarly, the approximation can be applied to the other dimension y

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{m,n} \approx \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

Finite Difference Approximation (contd...)

$$\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)_{m,n} \approx \frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2}+\frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2}$$

To model the steady state, no generation heat equation: $\nabla^2 T = 0$

This approximation can be simplified by specify $\Delta x = \Delta y$

and the nodal equation can be obtained as

$$T_{m+1,n}+T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0$$

This equation approximates the nodal temperature distribution based on the heat equation. This approximation is improved when the distance between the adjacent nodal points is decreased:

Since $\lim(\Delta x \to 0)\dfrac{\Delta T}{\Delta x}=\dfrac{\partial T}{\partial x}, \lim(\Delta y \to 0)\dfrac{\Delta T}{\Delta y}=\dfrac{\partial T}{\partial y}$

A System of Algebraic Equations

- The nodal equations derived previously are valid for all interior points satisfying the steady state, no generation heat equation. For each node, there is one such equation.

For example: for nodal point m=3, n=4, the equation is

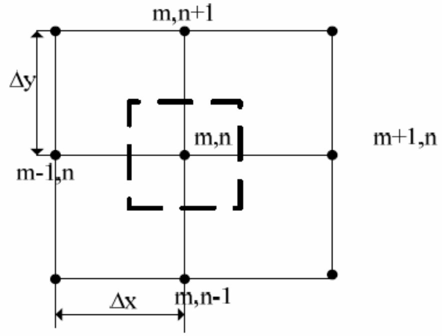
$T_{2,4} + T_{4,4} + T_{3,3} + T_{3,5} - 4T_{3,4} = 0$

$T_{3,4} = (1/4)(T_{2,4} + T_{4,4} + T_{3,3} + T_{3,5})$

- Nodal relation table for exterior nodes (boundary conditions) can be found in standard heat transfer textbooks (Table 4.2 in this presentation).
- Derive one equation for each nodal point (including both interior and exterior points) in the system of interest. The result is a system of N algebraic equations for a total of N nodal points.

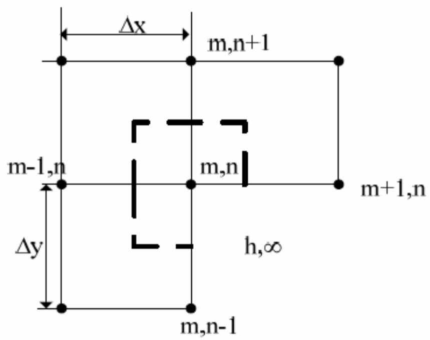
Table 4.2 Summary of nodal finite-difference methods

Configuration	Finite-Difference equations for Δx=Δy



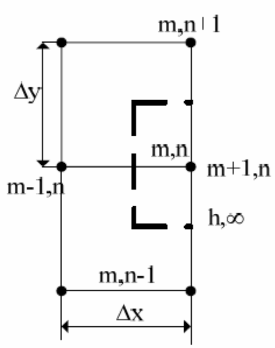
$$T_{m,n+1}+T_{m,n-1}+T_{m+1,n}+T_{m-1,n}-4T_{m,n}=0$$

Case 1. Interior node



$$2(T_{m-1,n}+T_{m,n+1})+(T_{m+1,n}+T_{m,n-1})+2\frac{h\Delta x}{k}T_\infty-2\left(3+\frac{h\Delta x}{k}\right)T_{m,n}=0$$

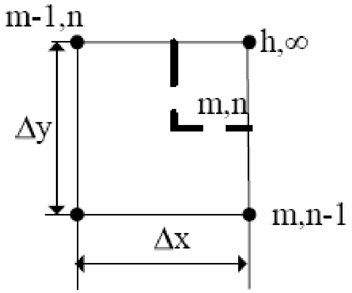
Case 2. Node at an internal corner with convection



$$2(T_{m-1,n}+T_{m,n+1}+T_{m,n-1})+2\frac{h\Delta x}{k}T_\infty-2\left(\frac{h\Delta x}{k}+2\right)T_{m,n}=0$$

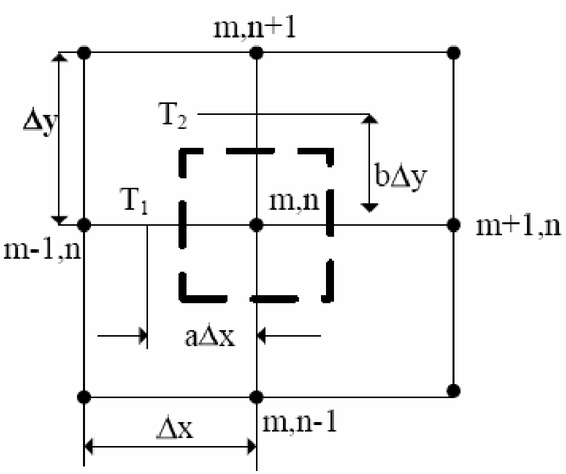
Case 3. Node at a plane surface with convection

Table 4.2 Summary of nodal finite-difference methods



$$2(T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + 2\frac{h\Delta x}{k}T_\infty - 2\left(\frac{h\Delta x}{k} + 1\right)T_{m,n} = 0$$

Case 4. Node at an external corner with convection



$$\frac{2}{a+1}T_{m+1,n} + \frac{2}{b+1}T_{m,n-1} + \frac{2}{a(a+1)}T_1 + \frac{2}{b)b+1)}T_2 - \left(\frac{2}{a} + \frac{2}{b}\right)T_{m,n} = 0$$

Case 5. Node near a curved surface maintained at a non uniform temperature

Matrix Form

The system of equations:

$$
\begin{aligned}
a_{11}T_1 &+ a_{12}T_2 + \cdots + a_{1N}T_N = C_1 \\
a_{21}T_1 &+ a_{22}T_2 + \cdots + a_{2N}T_N = C_2 \\
\vdots \quad &\quad \vdots \quad\quad \vdots \quad \vdots \quad\quad\quad \vdots \\
a_{N1}T_1 &+ a_{N2}T_2 + \cdots + a_{NN}T_N = C_N
\end{aligned}
$$

A total of N algebraic equations for the N nodal points and the system can be expressed as a matrix formulation: [**A**][**T**]=[**C**]

$$
where\ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}, T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix}, C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}
$$

Numerical Solutions

- Matrix form: [**A**][**T**]=[**C**].

From linear algebra: $[\mathbf{A}]^{-1}[\mathbf{A}][\mathbf{T}]=[\mathbf{A}]^{-1}[\mathbf{C}]$, $[\mathbf{T}]=[\mathbf{A}]^{-1}[\mathbf{C}]$

where $[\mathbf{A}]^{-1}$ is the inverse of matrix [**A**]. [**T**] is the solution vector.

- Matrix inversion requires cumbersome numerical computations and is not efficient if the order of the matrix is high (>10)

Numerical Solutions (contd…)

- Gauss elimination method and other matrix solvers are usually available in many numerical solution package. For example, "Numerical Recipes" by Cambridge University Press or their web source at www.nr.com.

- For high order matrix, iterative methods are usually more efficient. The famous Jacobi & Gauss-Seidel iteration methods will be introduced in the following.

Iteration

General algebraic equation for nodal point:

$$\sum_{j=1}^{i-1} a_{ij} T_j + a_{ii} T_i + \sum_{j=i+1}^{N} a_{ij} T_j = C_i ,$$

$$(\text{Example}: a_{31} T_1 + a_{32} T_2 + a_{33} T_3 + \mathrm{L} \ + a_{1N} T_N = C_1 ,\ i = 3)$$

Rewrite the equation of the form:

Replace (k) by (k-1) for the Jacobi iteration

$$T_i^{(k)} = \frac{C_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} T_j^{\boxed{(k)}} - \sum_{j=i+1}^{N} \frac{a_{ij}}{a_{ii}} T_j^{(k-1)}$$

Iteration (contd…)

- (k) - specify the level of the iteration, (k-1) means the present level and (k) represents the new level.
- An initial guess (k=0) is needed to start the iteration.
- By substituting iterated values at (k-1) into the equation, the new values at iteration (k) can be estimated
- The iteration will be stopped when $\max\left| Ti^{(k)} - Ti^{(k-1)} \right| \leq \varepsilon$, where ε specifies a predetermined value of acceptable error