MODULE 2: Worked-out Problems

Problem 1:

The steady-state temperature distribution in a one–dimensional slab of thermal conductivity 50W/m.K and thickness 50 mm is found to be T= $a+bx^2$, where a=200^0C, b=-2000^0C/ m^2, T is in degrees Celsius and *x* in meters.

(a) What is the heat generation rate in the slab?

(b) Determine the heat fluxes at the two wall faces. From the given temperature distribution and the heat fluxes obtained, can you comment on the heat generation rate?

Known: Temperature distribution in a one dimensional wall with prescribed thickness and thermal conductivity.

Find: (a) the heat generation rate, q in the wall, (b) heat fluxes at the wall faces and relation to q.

Schematic:

Assumptions: (1) steady-state conditions, (2) one –dimensional heat flow, (3) constant properties.

Analysis: (a) the appropriate form of heat equation for steady state, one dimensional condition with constant properties is

$$\dot{q} = -K\frac{d}{dx}\left[\frac{dT}{dx}\right]$$

$$\dot{q} = -k\frac{d}{dx}\left[\frac{d}{dx(a+bx^2)}\right] = -k\frac{d}{dx}[2bx] = -2bk$$

$$\dot{q} = -2(-2000^0\,CC/m^2)\times 50W/m.K = 2.0\times 10^5\,W/m^3$$

(b) The heat fluxes at the wall faces can be evaluated from Fourier’s law,

K=50W/m.K,q

q”$_x$(0)

$$q''_x(x) = -k\frac{dT}{dx}\bigg|_x$$

Using the temperature distribution T(x) to evaluate the gradient, find

$$q_x''(x) = -k\frac{d}{dx}[a+bx^2] = -2kbx.$$

The flux at the face, is then x=0

$$q''_x(0) = 0$$

$$\text{at}X = L, q''_x(l) = -2kbL = -2\times 50W/m.K(-2000^0 C/m^2)\times 0.050m$$

$$q''_x(L) = 10{,}000 W/m^2$$

Comments: from an overall energy balance on the wall, it follows that

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = 0 \qquad q''_x(0) - q''_x(L) + \dot{q}L = 0$$

$$\dot{q} = \frac{q''_x(L) - q''_x(0)}{L} = \frac{10{,}00w/m^2 - 0}{0.050m} = 2.0\times 10^5 W/m^3$$

Problem 2:

Consider a solar pond having three distinct layers of water-salt solution. The top and bottom layers are well mixed with salt. These layers are subjected to natural convention, but the middle layer is stationary. With this arrangement, the top and bottom surfaces of the middle layer is maintained at uniform temperature T_1 and T_2, where $T_1>T_2$. Solar radiation is absorbed in the middle layer in the form $q=Ae^{-mx}$, resulting in the following temperature distribution in the central layer

$$T(x) = -\frac{A}{ka^2}e^{-mx} + Bx + C$$

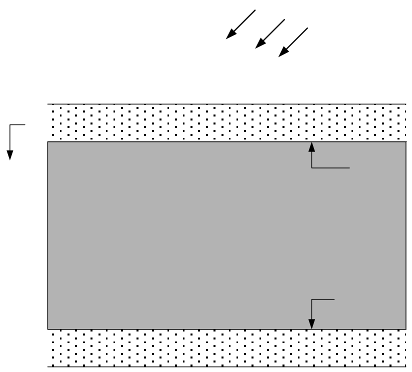
In the above equation, k is the thermal conductivity, and the constants A (W/m^3), a (1/m), B (K/m) and C(K) are also known.

Obtain expressions for the interfacial heat flux from the bottom layer to the middle layer, and from the middle layer to the top layer. Are the conditions are steady or transient? Next, obtain an expression for the rate at which thermal energy is generated in the entire middle layer, per unit surface area.

Known: Temperature distribution and distribution of heat generation in central layer of a solar pond.

Find: (a) heat fluxes at lower and upper surfaces of the central layer, (b) whether conditions are steady or transient (c) rate of thermal energy generation for the entire central layer.

Schematic:



Assumptions: (1) central layer is stagnant, (2) one-dimensional conduction, (3)constant properties.

Analysis (1) the desired fluxes correspond to conduction fluxes in the central layer at the lower and upper surfaces. A general form for the conduction flux is

$$q''_{cond} = \left[-k\frac{A}{km}e^{-mx+}B \right]$$

Hence

$$q''_l = q''_{cond(x=L0} = \left[-k\frac{A}{km}e^{-ml} + B \right] q''_u = q''_{cond(x=0)} = -k\left[\frac{A}{km} + B \right]$$

(b) Conditions are steady if ∂T/∂t=0. Applying the heat equation,

$$\frac{\partial^2 T}{\partial^2 t} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad -\frac{A}{k}e^{-mx} + \frac{A}{k}e^{-mx} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Hence conditions are steady since

$$\frac{\partial T}{\partial t} = 0 \quad \text{(for all 0<=x<=L)}$$

For the central layer, the energy generation is

$$\dot{E}''_g = \int_0^L q dx = A\int_0^L e - mxdx$$

$$\dot{E}_g = -\frac{A}{a}e^{-mx}\Big|_0^L = -\frac{A}{a}(e^{-mL} - 1) = \frac{A}{a}(1 - e^{-mL})$$

Alternatively, from an overall energy balance,

$$q''_2 - q''_1 + \dot{E}_g = 0 \qquad \dot{E}''_g = q''_1 - q''_2 = (-q''_{cond(x=0)}) - (q''_{cond(x=L)})$$

$$\dot{E}_g = k\frac{A}{km} + B - K\frac{A}{km}e^{-mL} + B = \frac{A}{m}(1 - e^{-mL})$$

Comments: Conduction is the negative x-direction, necessitating use of minus signs in the above energy balance.

Problem 3:

Consider 1D heat transfer across a slab with thermal conductivity k and thickness L. The steady state temperature is of the form $T=Ax^3+Bx^2+Cx+D$. Find expressions for the heat generation rate per unit volume in the slab and heat fluxes at the two wall faces (i.e. x=0, L).

Known: steady-state temperature distribution in one-dimensional wall of thermal conductivity, $T(x)=Ax^3+Bx^2+CX+d$.

Find: expressions for the heat generation rate in the wall and the heat fluxes at the two wall faces(x=0, L).

Assumptions: (1) steady state conditions, (2) one-dimensional heat flow, (3) homogeneous medium.

Analysis: the appropriate form of the heat diffusion equation for these conditions is

$$\frac{d^2T}{dx^2}+\frac{\dot{q}}{k}=0 \qquad \text{Or} \qquad \dot{q}=-k\frac{d^2T}{dx^2}$$

Hence, the generation rate is

$$\dot{q}=-\frac{d}{dx}\left[\frac{dT}{dx}\right]=-k\frac{d}{dx}[3Ax^2+2Bx+C+0]$$

$$\dot{q}=-k[6Ax+2B]$$

which is linear with the coordinate x. The heat fluxes at the wall faces can be evaluated from Fourier's law,

$$q''_x=-k\frac{dT}{dx}=-k[3Ax^2+2Bx+C]$$

Using the expression for the temperature gradient derived above. Hence, the heat fluxes are:

Surface x=0; $q''_x(0)=-kC$

Surface x=L;

$q''_x(L) = -K\,[3AL^2+2BL+C]$

COMMENTS: (1) from an over all energy balance on the wall, find

$$\dot{E}_{in}-\dot{E}_{out}+\dot{E}_g=0$$

$$q''_x(0)-q''_x(L)=(-kC)-(-K)[3AL^2+2BL+C]+\dot{E}_g=0$$

$$\dot{E}''_g=-3AkL^2-2BkL$$

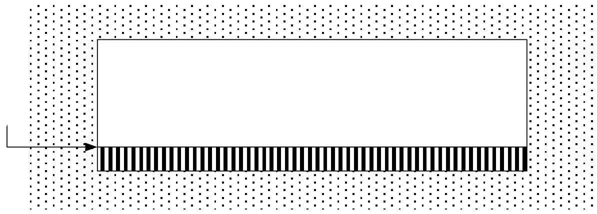
From integration of the volumetric heat rate, we can also find

$$\dot{E}''_g=\int_0^L \dot{q}(x)dx=\int_0^L -k[6Ax+2B]dx=-k[3Ax^2+2Bx]_0^L$$

$$\dot{E}''_g=-3AkL^2-2BkL$$

Problem 4:

Consider a one dimensional system of mass M with constant properties and no internal heat generation as shown in the figure below. The system is initially at a uniform temperature T_i. The electrical heater is suddenly switched ON, resulting in a uniform heat flux q''_o at the surface x=0 . The boundaries at x=L and elsewhere are perfectly insulated.



(a) Set up the differential equation along with the boundary and initial conditions for the temperature T(x,t).
(b) Sketch the temperature variation with x for the initial condition (t<=0) and for several times after the heater is switched ON. Comment whether a steady-state temperature distribution will ever be reached.
(c) For any given time, sketch the heat flux variation with x. Choose the following planes: x=0, x=L/2, and x=L.
(d) After time t_e, the heater power is switched off. Assuming no heat loss, derive an expression determine T_f , the final uniform temperature, as a function of the relevant parametersx

Known: one dimensional system, initially at a uniform temperature Ti, is suddenly exposed to a uniform heat flux at one boundary while the other boundary is insulated.

Find: (a) proper form of heat diffusion equation; identify boundary and initial conditions, (b) sketch temperature distributions for following conditions: initial condition (t<=0), several times after heater is energized ;will a steady-state condition be reached?, (c) sketch heat flux for x=0, L/2, L as a function of time, (d) expression for uniform temperature, Tf, reached after heater has been switched off the following an elapsed time , te, with the heater on.]

Schematic:

Assumptions: (1) one dimensional conduction, (2) no internal heat generation, (3) constant properties.

Analysis: (a) the appropriate form of the heat equation follows. Also the appropriate boundary and initial conditions are:

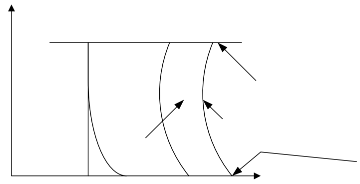
$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Initial condition: $T(x, 0) = T_i$ uniform temperature

Boundary conditions: x=0 $q''_o = -k\partial T/\partial x)_0 \; t > 0$

x=L $\qquad \partial T / \partial x)_L = 0$ Insulated

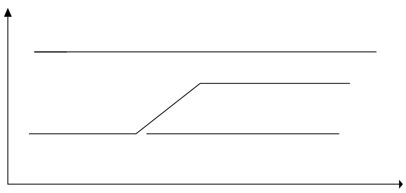
(b) The temperature distributions are as follows:



No steady-state condition will be reached since $\dot{E}_{in} - \dot{E}_{out}$ and $\dot{E}_{in}$ is constant.

(c) The heat flux as a function of time for positions x=0, L/2 and L appears as:

L



x

(d) If the heater is energized until $t=t_o$ and then switched off, the system will eventually reach a uniform temperature , T_f. Perform an energy balance on the system, for an interval of time $\Delta t=t_e$,

0

T =T

$$\dot{E}_{in} = \dot{E}_{st} \qquad E_{in} = Q_{in} = \int_0^{t_e} q''_0 A_s dt = q''_o A_s t_e \qquad E_{st} = Mc(T_f - T_i)$$

It follows that $q''_o A_s t_e = Mc(T_f - T_i)$ OR $T_f = T_i + \dfrac{q''_o A_s t_e}{Mc}$

q"0

o

0

Problem 5:

A 1–m-long metal plate with thermal conductivity k=50W/m.K is insulated on its sides. The top surface is maintained at 100^0C while the bottom surface is convectively cooled by a fluid at 20^0C. Under steady state conditions and with no volumetric heat generation, the temperature at the midpoint of the plate is measured to be 85^0C. Calculate the value of the convection heat transfer coefficient at the bottom surface.

Known: length, surface thermal conditions, and thermal conductivity of a Plate. Plate midpoint temperature.

Find: surface convection coefficient

Schematic:

Assumptions: (1) one-dimensional, steady conduction with no generation, (2) Constant properties

Analysis: for prescribed conditions, is constant. Hence,

$$q''_{cond} = \frac{T_1 - T_2}{L/2} = \frac{15^0 C}{0.5m/50W/m.k} = 1500W/m^2$$

$$q'' = \frac{T_1 - T_\infty}{(L/k) + (1/h)} = \frac{30^0 C}{(0.02 + 1/h)m^2.K/W} = 1500W/m^2$$

$$h = 30W/m^2.K$$

Comments: The contributions of conduction and convection to the thermal resistance are

$$R''_{t,cond} = \frac{L}{K} = 0.02m^2.K/W$$

$$R''_{t,cond} = \frac{1}{h} = 0.033m^2.K/W$$

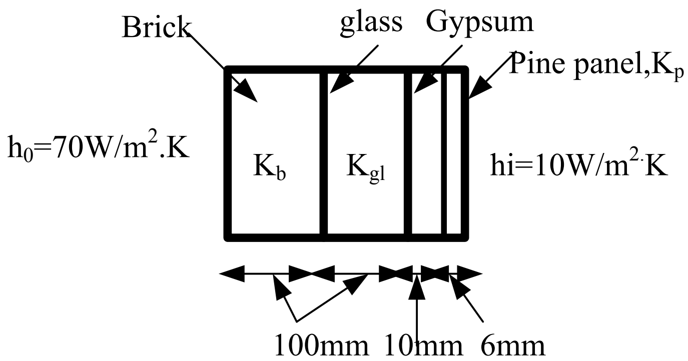
Problem 6:

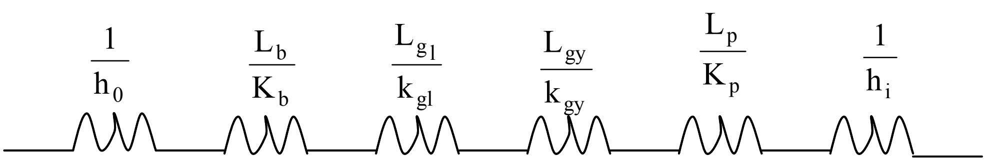
The wall of a building is a multi-layered composite consisting of brick (100-mm layer), a 100-mm layer of glass fiber(paper faced. $28kg/m^2$), a 10-mm layer of gypsum plaster (vermiculite), and a 6-mm layer of pine panel. If h_{inside} is $10W/m^2.K$ and $h_{outside}$ is $70W/m^2.K$, calculate the total thermal resistance and the overall coefficient for heat transfer.

Known: Material thickness in a composite wall consisting of brick, glass fiber, and vermiculite and pine panel. Inner and outer convection coefficients.

Find: Total thermal resistance and overall heat transfer coefficient.

Schematic:





Assumptions: (1) one dimensional conduction, (2) constant properties, (3) negligible contact resistance.

Properties: T= 300K: Brick, k_b=1.3 W/m.K: Glass fiber ($28kg/m^3$), k_{gl}= 0.038W/m.K: gypsum, k_{gy}=0.17W/m.K: pine panel, k_p=0.12W/m.K.

Analysis: considering a unit surface Area, the total thermal resistance

$$R''_{tot} = \frac{1}{h_0} + \frac{L_B}{K_B} + \frac{L_{g1}}{k_{g1}} + \frac{L_{gy}}{k_{gy}} + \frac{L_p}{K_p} + \frac{1}{h_i}$$

$$R''_{tot} = \left[\frac{1}{70} + \frac{0.1}{1.3} + \frac{0.1}{0.038} + \frac{0.01}{0.17} + \frac{0.006}{0.12} + \frac{1}{10}\right]\frac{m^2.K}{W}$$

$$R''_{tot} = (0.0143 + 0.0769 + 2.6316 + 0.0588 + 0.0500 + 0.1)m^2.K/W$$

$$R''_{tot} = 2.93m^2..K/W$$

The overall heat transfer coefficient is

$$U = \frac{1}{R_{tot}A} = \frac{1}{R''_{tot}} = (2.93m^2.K/W)^{-1}$$

$$U = 0.341W/m^2.K.$$

Comments: an anticipated, the dominant contribution to the total resistance is made by the insulation.

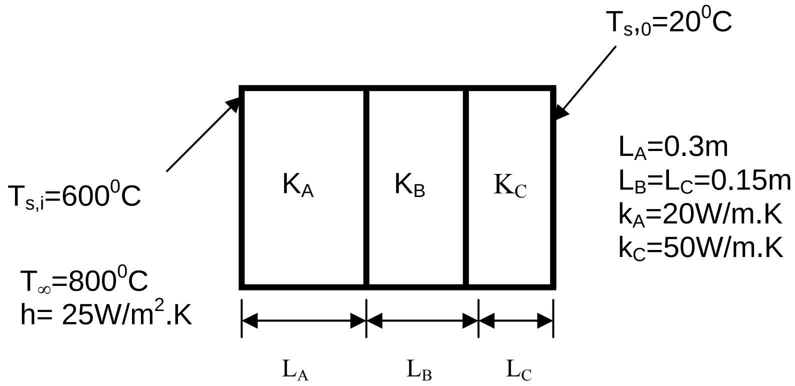
Problem 7:

The wall of an oven is a composite of the following layers. Layers A has a thermal conductivity k_A=20W/m.K, and layer C has a thermal conductivity k_C=50W/m.K. The corresponding thicknesses are L_A=0.30m and L_C=0.15m, respectively. Layer B is sandwiched between layers A and C, is of known thickness, L_B=0.15m, but unknown thermal conductivity k_B. Under steady-state operating conditions, the outer surface temperature is measured to be $T_{s,0}$=200C. Measurements also tell us that the inner surface temperature $T_{s,i}$ is 600^0C and the oven air temperature is T =800^0C. The inside convection coefficient h is known to be 25W/m^2.K. Find the value of k_B.

Known: Thickness of three material which form a composite wall and thermal conductivities of two of the materials. Inner and outer surface temperatures of the composites; also, temperature and convection coefficient associated with adjoining gas.

Find: value of unknown thermal conductivity, k_B.

Schematic:



Assumptions: (1) steady state conditions, (2) one-dimensional conduction, (3) constant properties, (4) negligible contact resistance, (5) negligible radiation effects.

Analysis: Referring to the thermal circuit, the heat flux may be expressed as

$$q'' = \frac{T_{s,i} - T_{s,0}}{\frac{L_A}{K_A} + \frac{L_B}{K_B} + \frac{L_C}{K_C}} = \frac{(600-20)^0 C}{\frac{0.3m}{0.018} + \frac{0.15m}{K_B} + \frac{0.15m}{50W/m.K}}$$

$$= \frac{580}{0.018 + 0.15/K_B} W/m^2$$

The heat flux can be obtained from

$$q'' = h(T_\infty - T_{s,i}) = 25W/m^2.K(800-600)^0 C$$
$$q'' = 5000 W/m^2$$

Substituting for heat flux,

$$\frac{0.15}{K_B} = \frac{580}{q''} - 0.018 = \frac{580}{5000} - 0.018 = 0.098$$
$$K_B = 1.53 W/m.K.$$

Comments: In an over, radiation effects are likely to have a significant influence on the net heat flux at the inner surface of the oven.

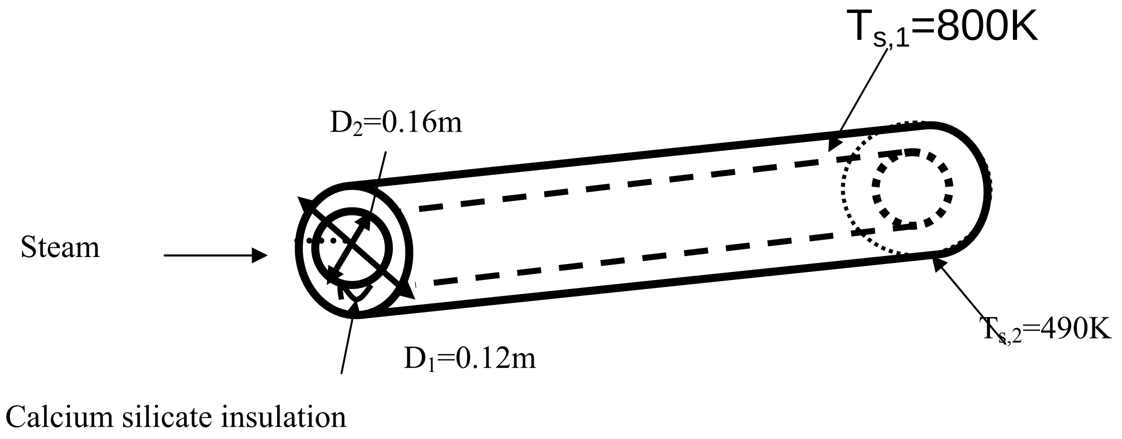
Problem 8:

A steam pipe of 0.12 m outside diameter is insulated with a 20-mm-thick layer of calcium silicate. If the inner and outer surfaces of the insulation are at temperatures of $T_{s,1}$=800 K and $T_{s,2}$=490 K, respectively, what is the heat loss per unit length of the pipe?

Known: Thickness and surface temperature of calcium silicate insulation on a steam pipe.

Find: heat loss per unit pipe length.

Schematic:



Assumptions: (steady state conditions, (2) one-dimensional conduction, (3) constant properties.

Properties: calcium silicate (T=645K): k=0.089W/m.K

Analysis: The heat per unit length is

$$q_r' = \frac{q_r}{q_L} = \frac{2\pi K(T_{s,1} - T_{s,2})}{\ln(D_2 / D_1)}$$

$$q_r' = \frac{2\pi(0.089\,W/m.K)(800 - 490)K}{\ln(0.16m/0.12m)}$$

$$q_r' = 603\,W/m$$

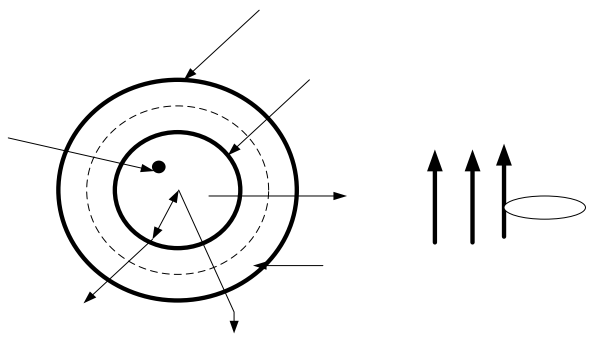
Comments: heat transferred to the outer surface is dissipated to the surroundings by convection and radiation.

Problem 9:

A cylindrical nuclear fuel rod of 0.1m dia has a thermal conductivity k=0.0W/m.K and generates uniformly 24,000W/m^3. This rod is encapsulated within another cylinder having an outer radius of 0.2m and a thermal conductivity of 4W/m.K. The outer surface is cooled by a coolant fluid at 100^{0}C, and the convection coefficient between the outer surface and the coolant is 20W/m^2.K. Find the temperatures at the interface between the two cylinders and at the outer surface.

Known: A cylindrical rod with heat generation is cladded with another cylinder whose outer surface is subjected to a convection process.
Find: the temperature at the inner surfaces, $T_{1,}$ and at the outer surface, T_c.

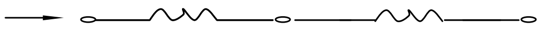
Schematic:



Assumptions: (1) steady-state conditions, (2) one-dimensional radial conduction, (3), negligible contact resistance between the cylinders.

Analysis: The thermal circuit for the outer cylinder subjected to the convection process is

q_1=24,000W/m^3
k_1=0.5W/m.K



$$R_1' = \frac{\ln r_o / r_1}{2\pi k_2}$$

$$R_2' = \frac{1}{h 2\pi r_o}$$

Using the energy conservation requirement, on the Inner cylinder,

$$\dot{E}_{out} = \dot{E}_g$$

Find that

$$q' = \dot{q}_1 \times \pi r_1^2$$

The heat rate equation has the form $\dot{q} = \Delta T / R'$, hence

$$T_i - T_\infty = q' \times (R_1' \times R_2') \text{and} q' = \Delta T / R'$$

Numerical values:

$$\begin{aligned} R_1' &= \ln 0.2/0.1/2\pi \times 4 W/m.K = 0.0276 K.m/W \\ R_2' &= 1/20 W/m^2.K \times 2\pi \times 0.20 m = 0.0398 K.m/W \\ q' &= 24{,}000 W/m^3 \times \pi \times (0.1)^2 m^2 = 754.0 W/m \end{aligned}$$

Hence

$$T_i = 100^0 C + 754.0 W/m \times (0.0276 + cccc) K.m/W = 100 + 50.8 = 150.8^0 C$$

$$T_C = 100^0 C + 754.0 W/m \times 0.0398 K.m/W = 100 + 30 = 130^0 C$$

Comments: knowledge of inner cylinder thermal conductivity is not needed.

Problem 10:

A steel cable having a diameter of 0.005m and an electrical resistance of $6*10^{-4}$ /m carries an electrical current of 700 A. The surrounding temperature of the cable is 300°C, and the effective coefficient associated with heat loss by convection and radiation between the cable and the environment is approximately $25W/m^2.K$.

(a) If the cable is uncoated, what is its surface temperature?

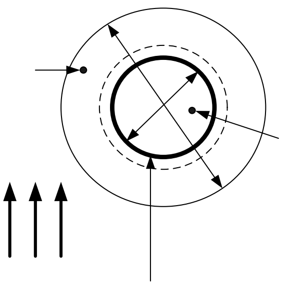
(b) If a very thin coating of electrical insulation is applied to the cable, with a contact resistance of $0.02m^2K/W$, what are the insulation and cable surface temperatures?

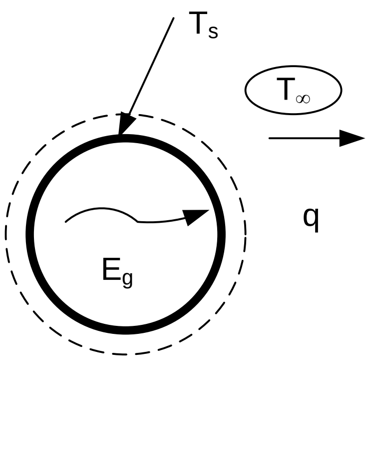
(c) What thickness of this insulation (k=0.5W/m.K) will yields the lowest value of the maximum insulation temperature? What is the value of the maximum temperature when the thickness is used?

Known: electric current flow, resistance, diameter and environmental conditions associated with a cable.

Find: (a) surface temperature of bare cable, (b) cable surface and insulation temperatures for a thin coating of insulation, (c) insulation thickness which provides the lowest value of the maximum insulation temperature. Corresponding value of this temperature.

Schematic:





Assumptions: (1) steady-state conditions, (2) one-dimensional conduction in r, (3) constant properties.

Analysis: (a) the rate at which heat is transferred to the surroundings is fixed by the rate of heat generation in the cable. Performing an energy balance for a control surface about the cable, it follows that $\dot{E}_g = q$ or, for the bare cable, $I^2R_e^{'}L = h(\pi D_i L)(T_s - T_\infty).\text{with}q^{'} = I^2R_e^{'} = (700A)^2(6\times10^{-4}\Omega/m) = 294W/m.$

It follows that

$$T_s = T_\infty + \frac{q^{'}}{h\pi D_i} = 30^0C + \frac{294W/m}{(25W/m^2.K)\pi(0.005m)}$$

$$T_s = 778.7^0C$$

(b) With thin coating of insulation, there exists contact and convection resistances to heat transfer from the cable. The heat transfer rate is determined by heating within the cable, however, and therefore remains the same,

$$q = \frac{T_s - T_\infty}{R_{t,c} + \dfrac{1}{h\pi D_i L}} = \frac{T_s - T\infty}{\dfrac{R_{t,c}}{\pi D_i L} + \dfrac{1}{h\pi D_i L}}$$

$$q^{'} = \frac{\pi D_i(T_s - T_\infty)}{R_{t,c} + \dfrac{1}{h}}$$

And solving for the surface temperature, find

$$T_s = \frac{q^{'}}{\pi D_i}\left(R_{t,c} + \frac{1}{h}\right) + T_\infty = \frac{294W/m}{\pi(0.005m)}\left(0.02\frac{m^2.K}{W} + 0.04\frac{m^2.K}{W}\right) + 30^0C$$

$$T_s = 1153^0C$$

The insulation temperature is then obtained from

$$q = \frac{T_s - T_\infty}{R_{t,e}}$$

Or

$$T_i = T_s - qR_{t,c} = 1153^0C - q\frac{R^{''}_{t,c}}{\pi D_i L} = 1153^0C - \frac{294\dfrac{W}{m}\times 0.02\dfrac{m^2.K}{W}}{\pi(0.005m)}$$

$$T_i = 778.7^0C$$

(c) The maximum insulation temperature could be reduced by reducing the resistance to heat transfer from the outer surface of the insulation. Such a reduction is possible $D_i<D_{cr.}$

$$r_{cr} = \frac{k}{h} = \frac{0.5W/m.K}{25W/m^2.K} = 0.02m$$

Hence, $D_{cr} = 0.04m > D_i = 0.005m$. To minimize the maximum temperature, which exists at the inner surface of the insulation, add insulation in the amount.

$$t = \frac{D_0 - D_i}{2} = \frac{D_{cr} - D_i}{2} = \frac{(0.04 - 0.005)m}{2}$$

$$t = 0.0175m$$

The cable surface temperature may then be obtained from

$$q'' = \frac{T_s - T_\infty}{\frac{R''_{t,c}}{\pi D_i} + \frac{\ln(D_{c,r}/D_i)}{2\pi\pi} + \frac{1}{h\pi\pi_{c,r}}} = \frac{T_s - 30^0C}{\frac{0.02m^2.K/W}{\pi(0.005m)} + \frac{\ln(0.04/0.005)}{2\pi\pi(0.5W/.)} + \frac{1}{25\frac{W}{m^2.K}\pi(0.04m)}}$$

hence,

$$294\frac{W}{m} = \frac{T_s - 30^0C}{(1.27 + 0.66 + 0.32)m.K/W} = \frac{T_s - 30^0C}{2.25m.K/W}$$

$$T_s = 692.5^0C$$

recognizing that, $q = (T_s - T_i)/R_{t,c,}$

$$T_i = T_s - qR_{t,c,} = T_s - q\frac{R''_{t,c}}{\pi D_i L} = 692.5^0C - \frac{294\frac{W}{m} \times 0.02\frac{m^2.K}{W}}{\pi(0.005m)}$$

$$T_i = 318.2^0C$$

Comments: use of the critical insulation in lieu of a thin coating has the effect of reducing the maximum insulation temperature from 778.7^{0}C to 318.2^{0}C. Use of the critical insulation thickness also reduces the cable surface temperatures to 692.5^{0}C from 778.7^{0}C with no insulation or fro 1153^{0}C with a thin coating.