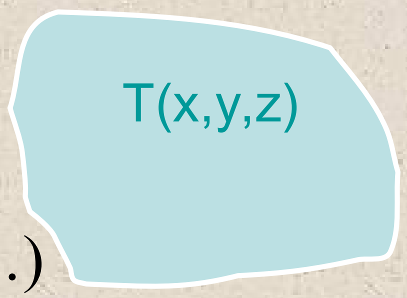
MODULE 2

One Dimensional Steady State Heat Conduction

Objectives of conduction analysis

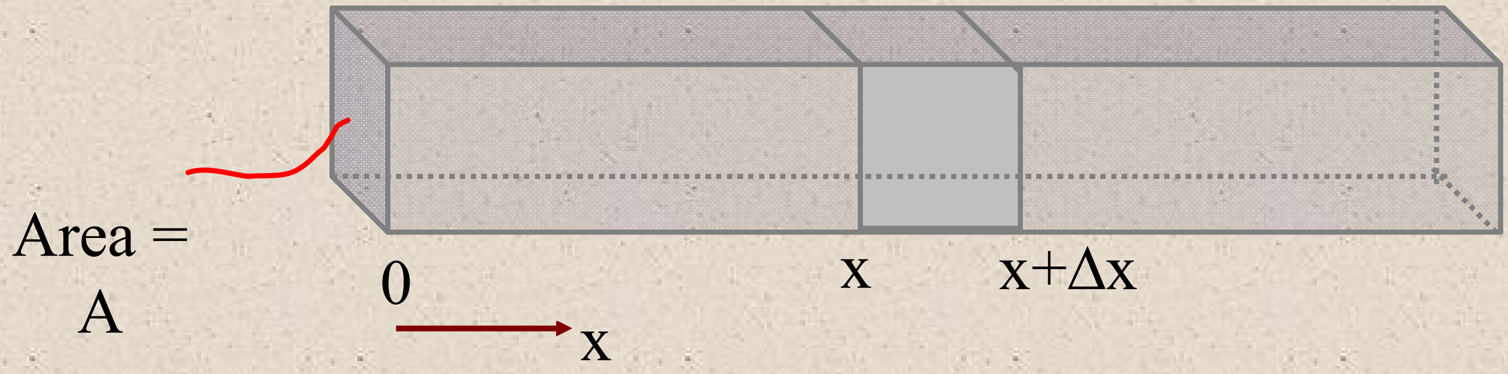
To determine the temperature field, T(x,y,z,t), in a body (i.e. how temperature varies with position within the body)

- T(x,y,z,t) depends on:
  - boundary conditions
  - initial condition
  - material properties (k, c^p, ρ …)
  - geometry of the body (shape, size)

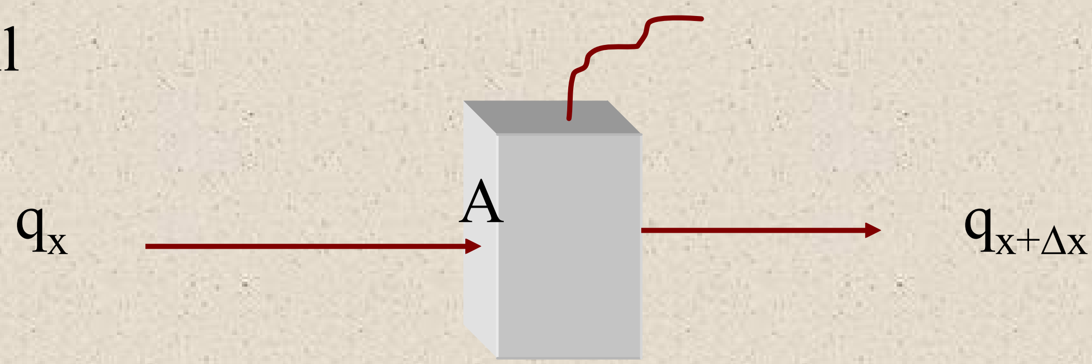


- Why we need T(x,y,z,t) ?
  - to compute heat flux at any location (using Fourier's eqn.)
  - compute thermal stresses, expansion, deflection due to temp. etc.
  - design insulation thickness
  - chip temperature calculation
  - heat treatment of metals

Unidirectional heat conduction (1D)



Solid bar, insulated on all long sides (1D heat conduction)



$\dot{q}$ = Internal heat generation per unit vol. (W/m^3)

Unidirectional heat conduction (1D)

First Law (energy balance) $$(\dot{E}_{in} - \dot{E}_{out}) + \dot{E}_{gen} = \dot{E}_{st}$$

$$q_x - q_{x+\Delta x} + A(\Delta x)\dot{q} = \frac{\partial E}{\partial t}$$

$$E = (\rho A \Delta x)u$$

$$\frac{\partial E}{\partial t} = \rho A \Delta x \frac{\partial u}{\partial t} = \rho A \Delta x c \frac{\partial T}{\partial t}$$

$$q_x = -kA\frac{\partial T}{\partial x}$$

$$q_{x+\Delta x} = q_x + \frac{\partial q_x}{\partial x}\Delta x$$

Unidirectional heat conduction (1D)(contd…)

$$-kA\frac{\partial T}{\partial x}+kA\frac{\partial T}{\partial x}+A\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)\Delta x+A\Delta x\dot{q}=\rho Ac\Delta x\frac{\partial T}{\partial t}$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\dot{q}=\rho c\frac{\partial T}{\partial t}$$

Longitudinal conduction

Internal heat generation

Thermal inertia

If k is a constant

$$\frac{\partial^2 T}{\partial x^2}+\frac{\dot{q}}{k}=\frac{\rho c}{k}\frac{\partial T}{\partial t}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Unidirectional heat conduction (1D)(contd…)

- For T to rise, LHS must be positive (heat input is positive)
- For a fixed heat input, T rises faster for higher α
- In this special case, heat flow is 1D. If sides were not insulated, heat flow could be 2D, 3D.

Boundary and Initial conditions:

- The objective of deriving the heat diffusion equation is to determine the temperature distribution within the conducting body.

- We have set up a differential equation, with T as the dependent variable. The solution will give us T(x,y,z). Solution depends on boundary conditions (BC) and initial conditions (IC).

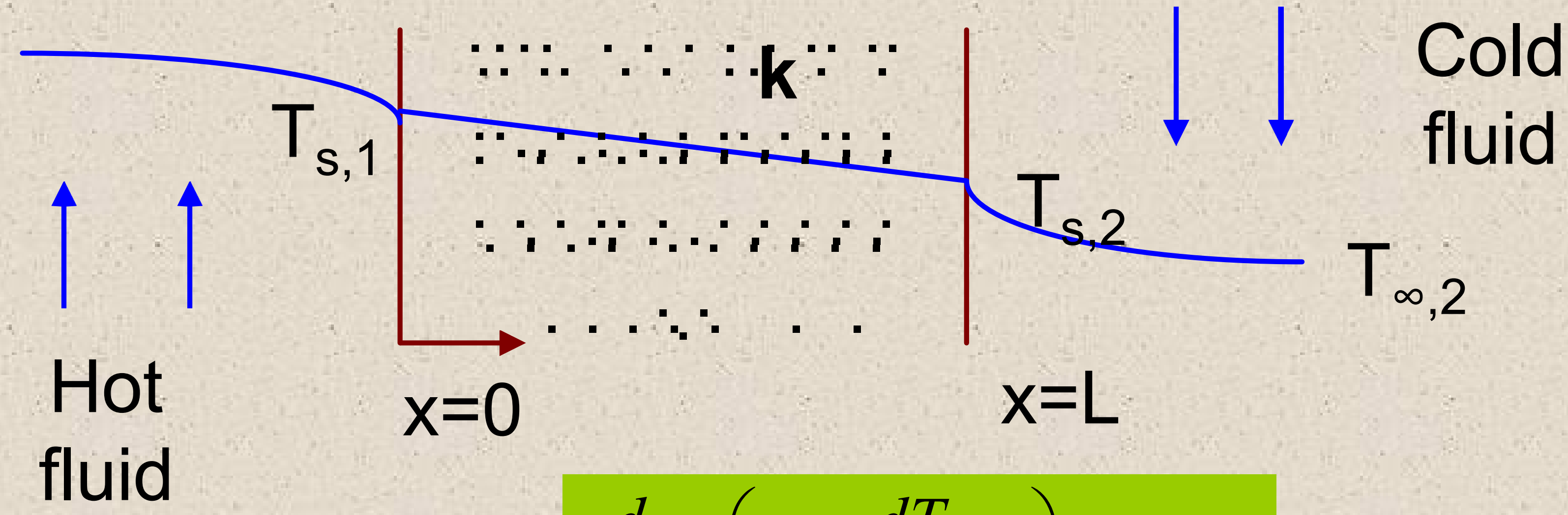
Boundary and Initial conditions (contd…)

How many BC's and IC's ?

- Heat equation is second order in spatial coordinate. Hence, 2 BC's needed for each coordinate.
  * 1D problem: 2 BC in x-direction
  * 2D problem: 2 BC in x-direction, 2 in y-direction
  * 3D problem: 2 in x-dir., 2 in y-dir., and 2 in z-dir.
- Heat equation is first order in time. Hence one IC needed

1- Dimensional Heat Conduction

The Plane Wall :



$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$$

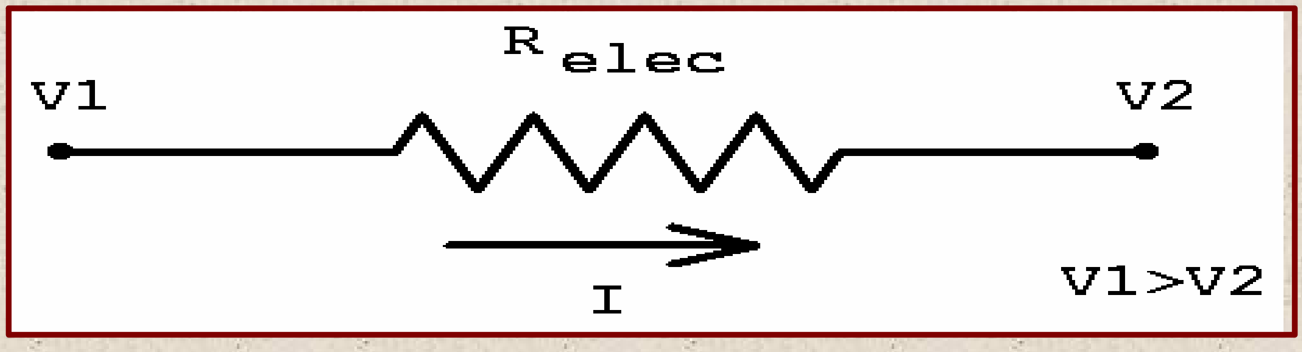
Const. K; solution is:

$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}\left(T_{s,1} - T_{s,2}\right) = \frac{T_{s,1} - T_{s,2}}{L / kA}$$

Thermal resistance (electrical analogy)

OHM's LAW :Flow of Electricity

$$V = IR_{elect}$$

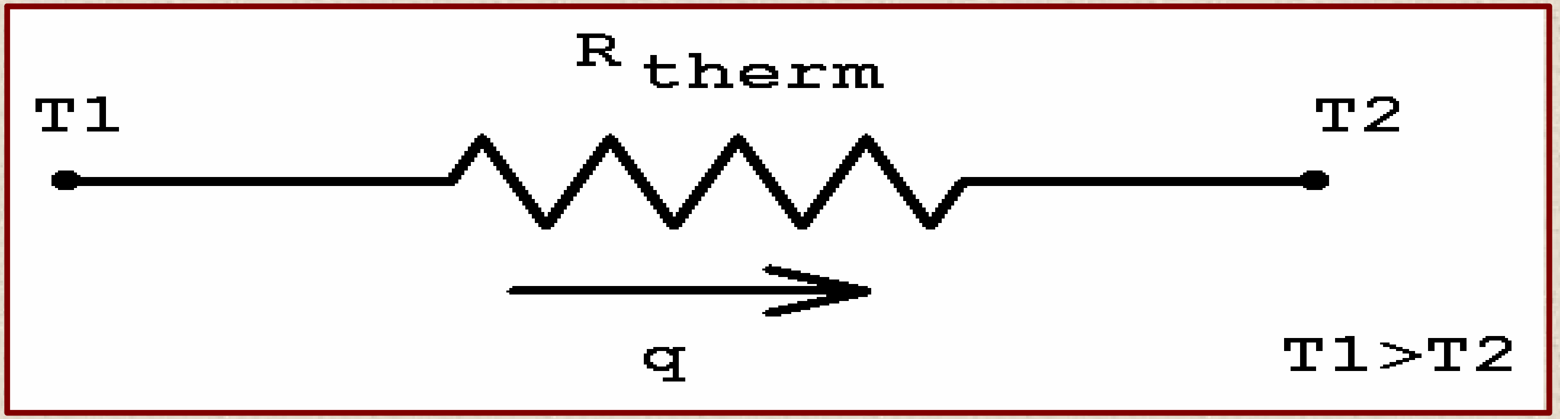


Voltage Drop = Current flow×Resistance

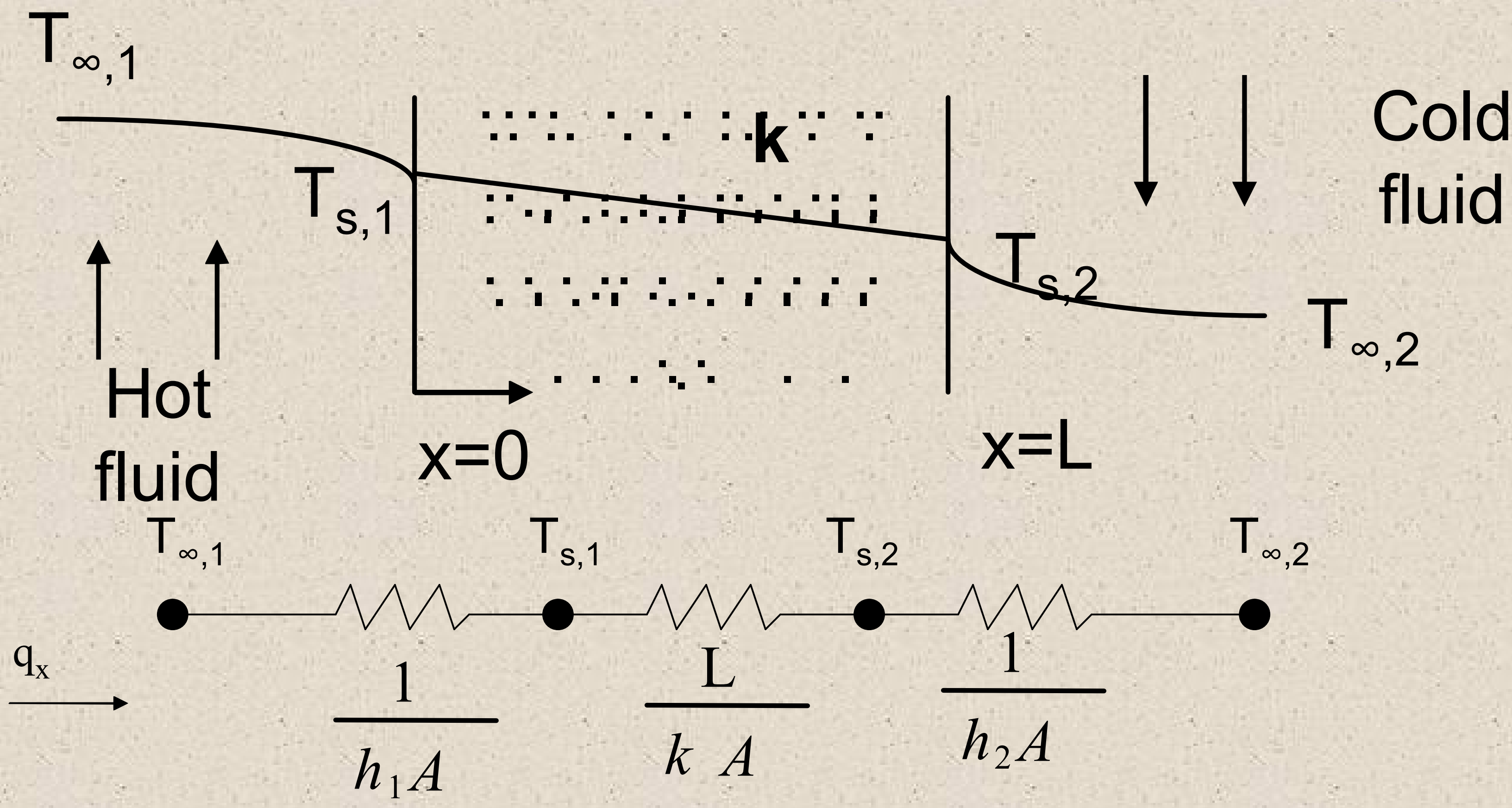
Thermal Analogy to Ohm's Law :

$$\Delta T = qR_{therm}$$

Temp Drop=Heat Flow×Resistance



1 D Heat Conduction through a Plane Wall



$$\sum R_t = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$ (Thermal Resistance)

Resistance expressions

THERMAL RESISTANCES

- Conduction

$$R_{cond} = \Delta x/kA$$

- Convection

$$R_{conv} = (hA)^{-1}$$

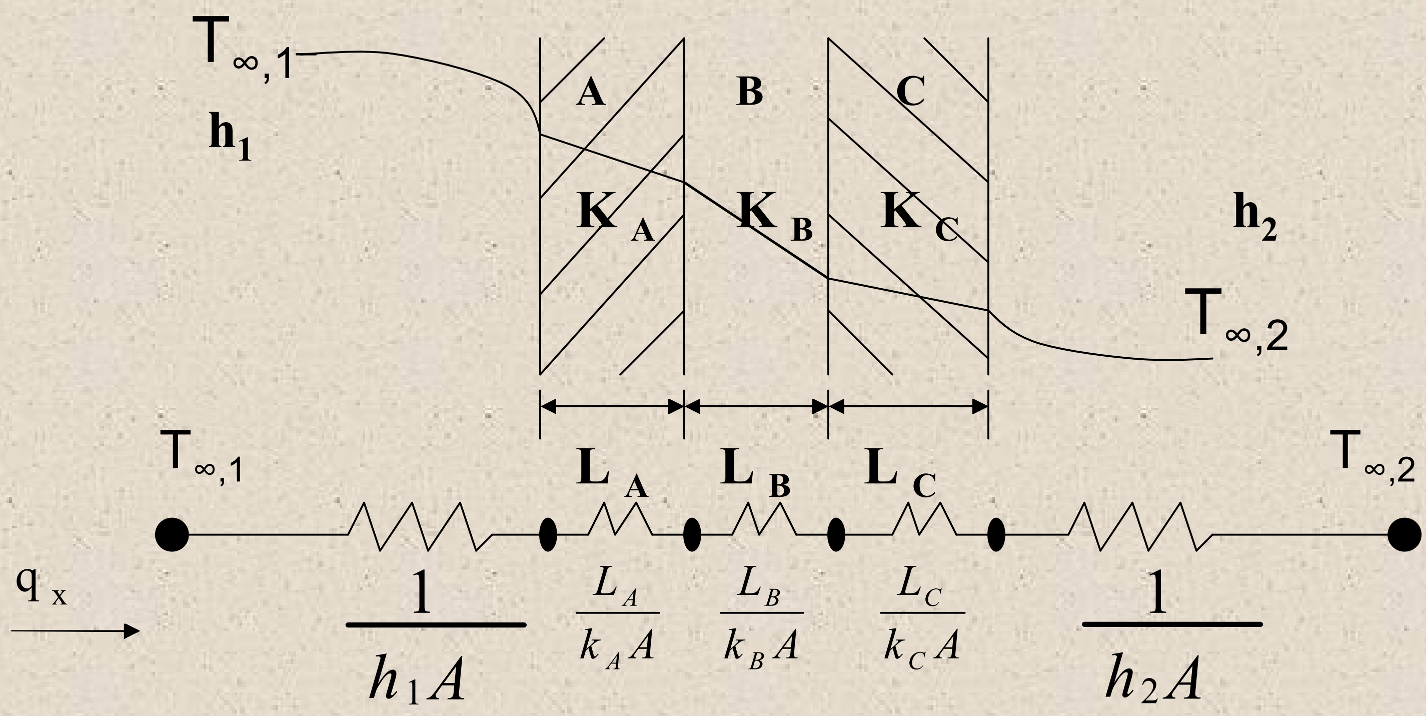
- Fins

$$R_{fin} = (h\eta A)^{-1}$$

- Radiation*(aprox)*

$$R_{rad} = [4A\sigma F(T_1T_2)^{1.5}]^{-1}$$

Composite Walls :



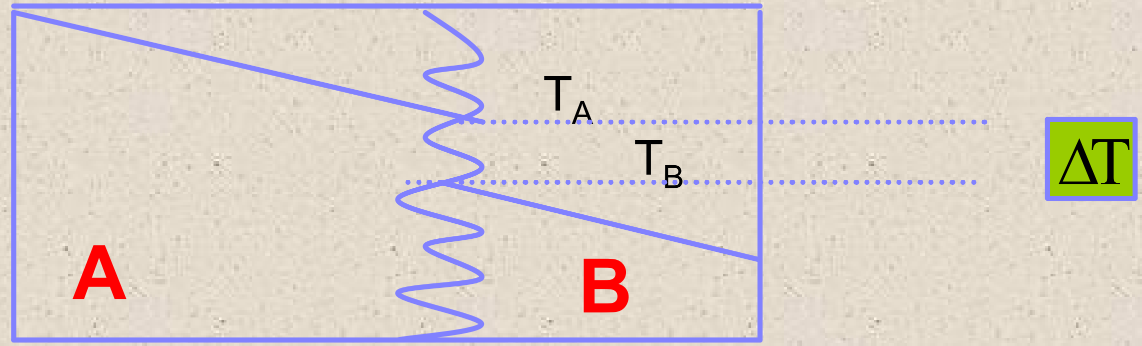
$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{\sum R_t} = \frac{T_{\infty,1} - T_{\infty,2}}{\frac{1}{h_1 A} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_2 A}} = UA\,\Delta T$$

$$where, U = \frac{1}{R_{tot} A} = \text{Overall heat transfer coefficient}$$

Overall Heat transfer Coefficient

$$U = \frac{1}{R_{total} A} = \frac{1}{\frac{1}{h_1} + \Sigma \frac{L}{k} + \frac{1}{h_2}}$$

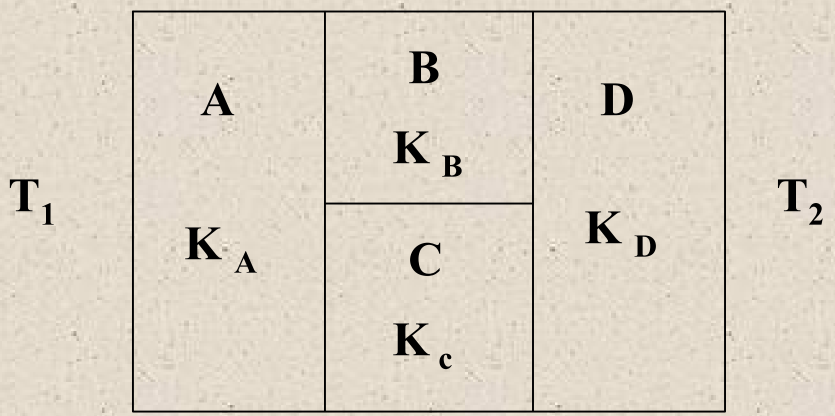
Contact Resistance :



$$R_{t,c} = \frac{\Delta T}{q_x}$$

$$U = \frac{1}{\frac{1}{h_1} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_2}}$$

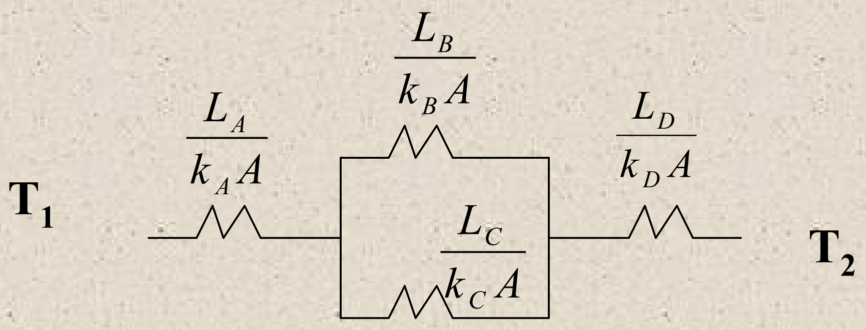
Series-Parallel :



$A_B + A_C = A_A = A_D$

$L_B = L_C$

Series-Parallel (contd…)



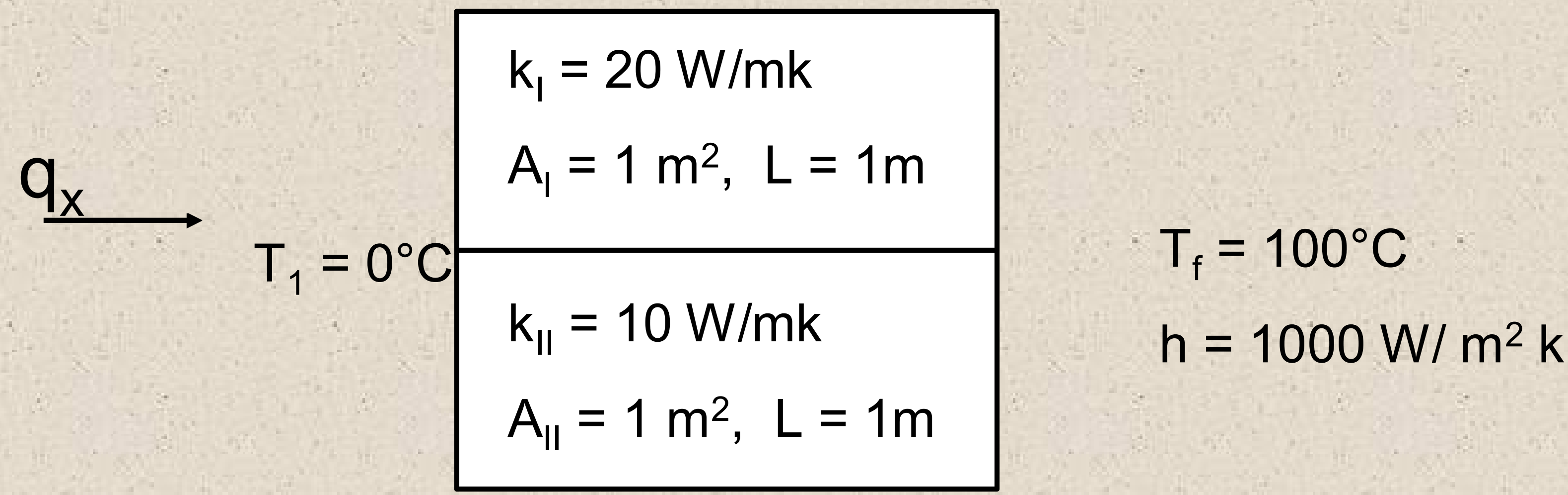
Assumptions :

(1) Face between B and C is insulated.

(2) Uniform temperature at any face normal to X.

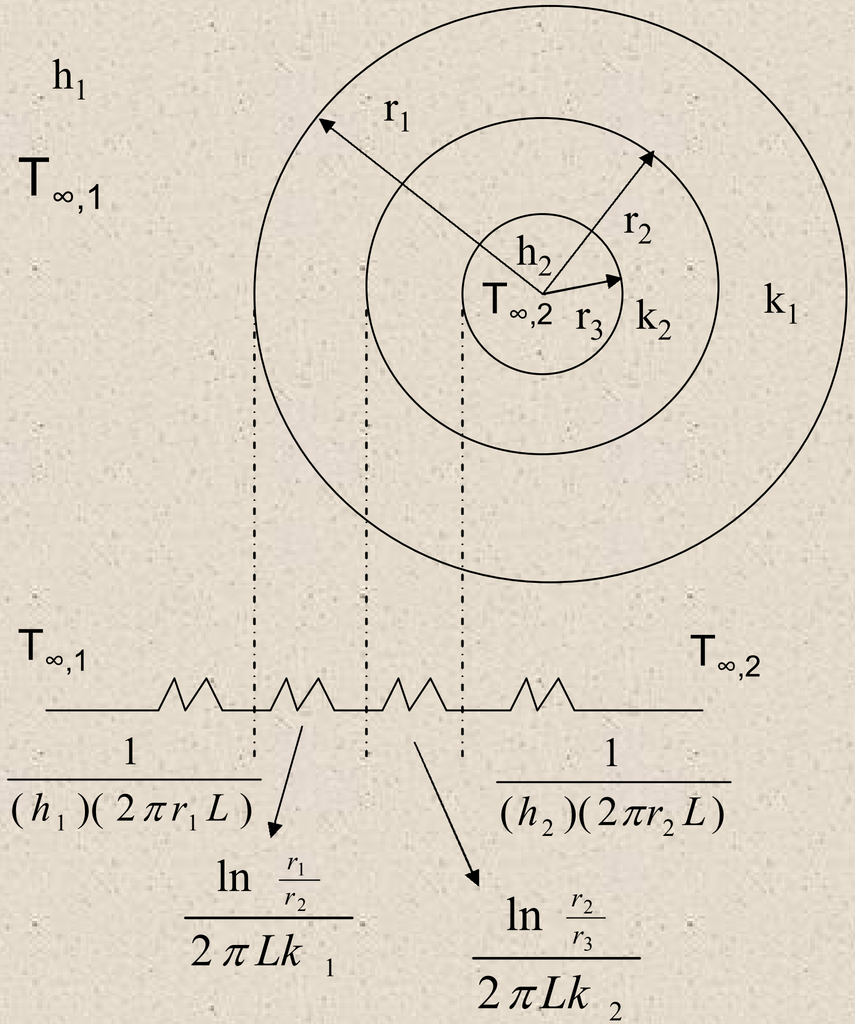
Example:

Consider a composite plane wall as shown:



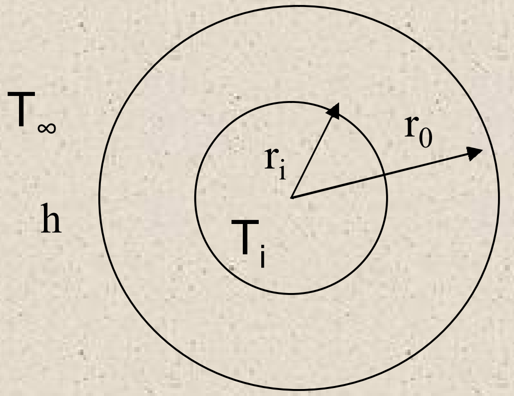
Develop an approximate solution for the rate of heat transfer through the wall.

1 D Conduction(Radial conduction in a composite cylinder)



$$q_r = \frac{T_{\infty,2} - T_{\infty,1}}{\sum R_t}$$

Critical Insulation Thickness :



Insulation Thickness : $r_o - r_i$

$$R_{tot} = \frac{\ln\left(\frac{r_0}{r_i}\right)}{2\pi kL} + \frac{1}{(2\pi r_0 L)h}$$

Objective : decrease q , increases R_{tot}

Vary r_0 ; as r_0 increases ,first term increases, second term decreases.

Critical Insulation Thickness (contd…)

Maximum – Minimum problem

Set $\dfrac{dR_{tot}}{dr_0} = 0$

$$\frac{1}{2\pi k r_0 L} - \frac{1}{2\pi h L r_0^2} = 0$$

$$r_0 = \frac{k}{h}$$

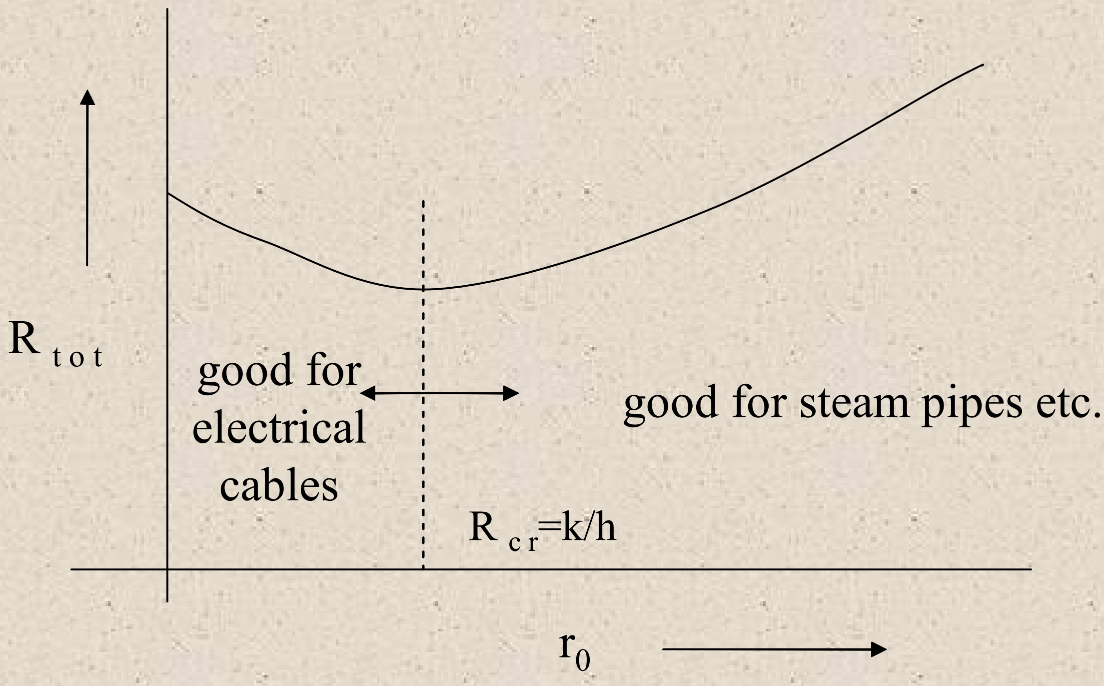
Max or Min. ? Take : $\dfrac{d^2 R_{tot}}{dr_0^2} = 0$ at $r_0 = \dfrac{k}{h}$

$$\frac{d^2 R_{tot}}{dr_0^2} = \frac{-1}{2\pi k r_0^2 L} + \frac{1}{\pi r_0^2 h L}\bigg|_{r_0 = \frac{k}{h}}$$

$$= \frac{h^2}{2\pi L k^3} \rangle 0$$

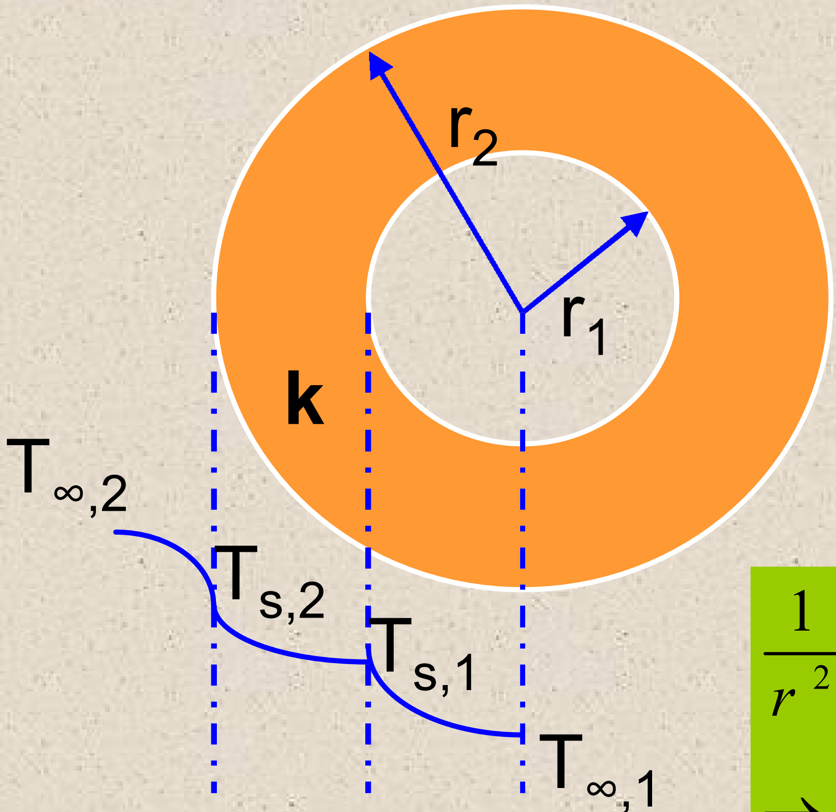
Critical Insulation Thickness (contd…)

Minimum q at $r_0 = (k/h) = r_{cr}$ (critical radius)



1D Conduction in Sphere

1D Conduction in Sphere



Inside Solid:

$$\frac{1}{r^2}\frac{d}{dr}\left(kr^2\frac{dT}{dr}\right)=0$$

$$\rightarrow T(r)=T_{s,1}-\{T_{s,1}-T_{s,2}\}\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$$

$$\rightarrow q_r=-kA\frac{dT}{dr}=\frac{4\pi k\left(T_{s,1}-T_{s,2}\right)}{\left(1/r_1-1/r_2\right)}$$

$$\rightarrow R_{t,cond}=\frac{1/r_1-1/r_2}{4\pi k}$$

Conduction with Thermal Energy Generation

$$\dot{q} = \frac{\dot{E}}{V}$$ = Energy generation per unit volume

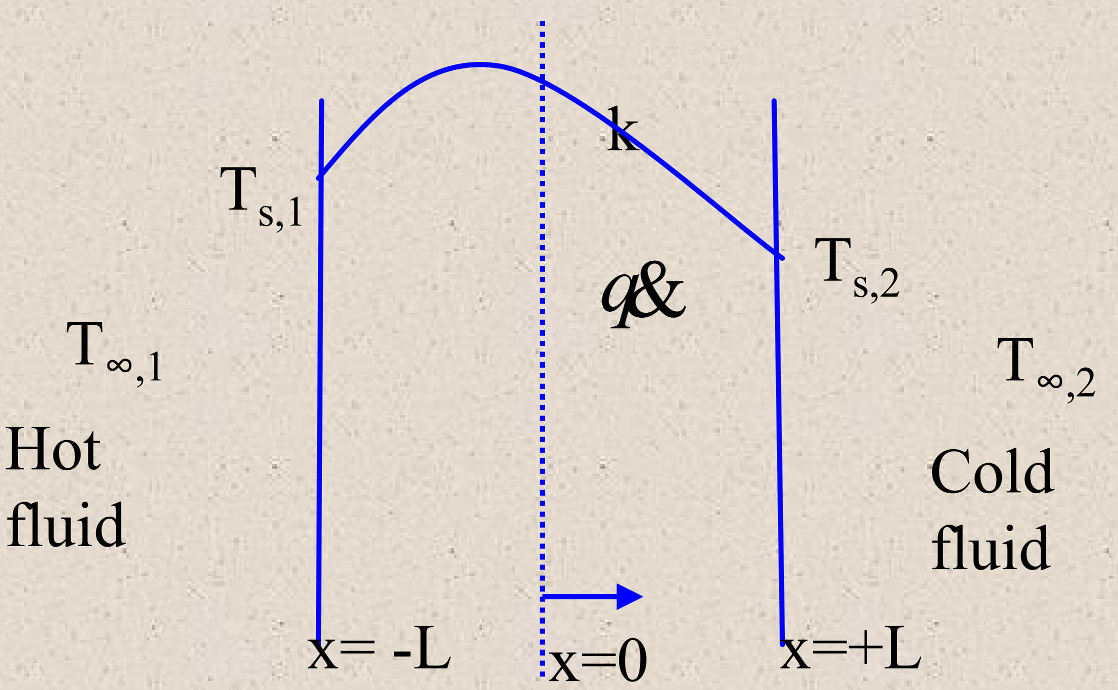
Applications: * current carrying conductors

* chemically reacting systems

* nuclear reactors

Conduction with Thermal Energy Generation

The Plane Wall :



Assumptions:

1D, steady state, constant k,

uniform $\dot{q}$

Conduction With Thermal Energy Generation (contd…)

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0$$

$$Boundary \quad cond.: \quad x = -L, \quad T = T_{s,1}$$

$$x = +L, \quad T = T_{s,2}$$

$$Solution: \quad T = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

Conduction with Thermal Energy Generation (cont..)

$$Use\ boundary\ conditions\ to\ find\ C_1\ and\ C_2$$

$$Final\ solution: T = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,2} + T_{s,1}}{2}$$

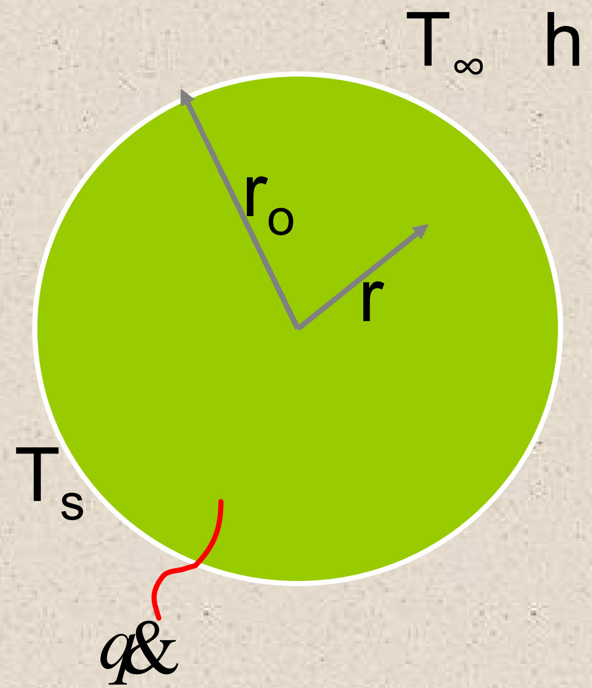
Not linear any more

$$Heat\ flux: \qquad q_x'' = -k\frac{dT}{dx}$$

Derive the expression and show that it is not independent of x any more

Hence thermal resistance concept is not correct to use when there is internal heat generation

Cylinder with heat source



Assumptions:

1D, steady state, constant k, uniform $\dot{q}$

Start with 1D heat equation in cylindrical co-ordinates:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{q}}{k}=0$$

Cylinder With Heat Source

$$Boundary\ cond.: \quad r = r_0, \quad T = T_s$$

$$r = 0, \quad \frac{dT}{dr} = 0$$

$$Solution: \quad T(r) = \frac{\dot{q}}{4k} r_0^2 \left(1 - \frac{r^2}{r_0^2}\right) + T_s$$

T_s may not be known. Instead, $T_{\propto}$ and h may be specified.

Exercise: Eliminate T_s, using $T_{\propto}$ and h.

Cylinder with heat source (contd…)

Example:

A current of 100A is passed through a stainless steel wire having a thermal conductivity K=25W/mK, diameter 3mm, and electrical resistivity R = 2.0 Ω. The length of the wire is 1m. The wire is submerged in a liquid at 100°C, and the heat transfer coefficient is 10W/m^{2}K. Calculate the centre temperature of the wire at steady state condition.