MODULE VII

TIME INTEGRATION TECHNIQUES: ASSIGNMENT

VII.1. Which of the following can be called an ***implicit method***?

a) Forward Euler method.
b) Backward Euler method.
c) Mid-point rule.
d) Crank-Nicolson method.

VII.2. For evaluation of value of $\phi(t)$ at $t = t_{n+1}$, explicit time integration methods

a) require value of $\phi(t)$ at some point $t > t_n$.
b) require an iterative solution.
c) do not require value of $\phi(t)$ at any point $t > t_n$.
d) directly obtain the solution without any iteration.

VII.3. Runge-Kutta methods for time integration are

a) predictor-corrector type methods.
b) two-point (or two-level) methods.
c) multi-level (or multi-point) methods.
d) always implicit methods.

VII.4. Consider the numerical algorithm for solution of unsteady generic transport equation which uses explicit Euler method for time integration, upwind scheme for convective term and central difference method for discretization of diffusive terms. This algorithm is

a) unconditionally stable.
b) unconditionally unstable.
c) conditionally stable.
d) unconditionally unstable if diffusion is zero.

VII.5. Consider the one-dimensional generic transport equation

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = \frac{\Gamma}{\rho}\frac{\partial^2 \phi}{\partial x^2}.$$

If we use the explicit Euler method for time integration, the discretized equation for ϕ_i^{n+1} on an uniform spatial grid can be written as

$$\phi_i^{n+1} = a_i\phi_i^n + a_{i+1}\phi_{i+1}^n + a_{i-1}\phi_{i-1}^n$$

where a_i are coefficients dependent on the material properties, spatial grid size, Δx, and time step Δt. These coefficients also depend on the spatial discretization scheme employed for the first and the second order spatial derivatives. Stability of the time integration requires that all a_i's must be positive.

(a) What are values of a_i's if central difference scheme (CDS) is used for the approximation of the first as well as the second order spatial derivative? Show that the resulting scheme would be stable if

$$\Delta t < \frac{\rho(\Delta x)^2}{2\Gamma} \text{ and } \frac{\rho u \Delta x}{\Gamma} < 2.$$

Will this scheme be stable if there is no diffusion?

(b) If the convective term is evaluated using the first order upwind differences, and CDS is used for the second order derivative, what would be the values of the coefficients a_i's? Show that the resulting scheme would be stable if

$$\Delta t < \frac{1}{\dfrac{2\Gamma}{\rho(\Delta x)^2} + \dfrac{u}{\Delta x}}.$$

Will this scheme be stable for negligible or no diffusion?

VII.6. Multipoint Adams methods for the initial value problem

$$\frac{d\phi}{dt} = f(\phi, t); \quad \phi(t_0) = \phi_0.$$

are derived by fitting a Lagrange polynomial to the derivative (i.e. to $f(\phi, t)$) at a number of points in time. Show that

a. the first order method is the implicit Euler method given by

$$\phi^{n+1} = \phi^n + \Delta t\, f(\phi^{n+1}, t_{n+1})$$

b. the second order method is the Crank-Nicolson method given by

$$\phi^{n+1} = \phi^n + \frac{\Delta t}{2}\,[f(\phi^n, t_n) + f(\phi^{n+1}, t_{n+1})],$$

c. the third order method is given by

$$\phi^{n+1} = \phi^n + \frac{\Delta t}{12}\,[5f(\phi^{n+1}, t_{n+1}) + 8f(\phi^n, t_n) - f(\phi^{n-1}, t_{n-1})].$$

Assume uniform time step Δt and use the following formulae in your derivation:

$$\int_{t_n}^{t_{n+1}} (t - t_n)dt = \Delta t^2/2, \quad \int_{t_n}^{t_{n+1}} (t - t_{n+1})dt = -\Delta t^2/2, \quad \int_{t_n}^{t_{n+1}} (t - t_n)(t - t_{n+1})dt = -\Delta t^3/6$$

$$\int_{t_n}^{t_{n+1}} (t - t_{n-1})(t - t_n)dt = 5\Delta t^3/6, \quad \int_{t_n}^{t_{n+1}} (t - t_{n-1})(t - t_{n+1})dt = -2\Delta t^3/3.$$

VII.7. Consider one-dimensional heat conduction equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

where α is thermal diffusivity. Central difference scheme is commonly used for spatial discretization, whereas a suitable time integration scheme --- explicit or implicit --- can be used for temporal discretization.

a. If we use the explicit Euler method for time integration, the discretized equation for T_i^{n+1} on an uniform spatial grid can be written as

$$T_i^{n+1} = a_i T_i^n + a_{i+1} T_{i+1}^n + a_{i-1} T_{i-1}^n$$

where a_i are coefficients dependent on the material properties, spatial grid size Δx, and time step Δt. What are the values of a_i's. Stability of the explicit time integration requires that all a_i's must be positive. What is the stability condition for this explicit scheme based on this criterion?

b. We can instead opt for an implicit scheme for sake of stability of time integration. Obtain the discretized equations using multi-point Adams-Moulton methods derived in Question 6, i.e. (i) Crank-Nicolson and (ii) third order Adams-Moulton method.

c. An implicit scheme of second order accuracy can also be obtained using a quadratic backward approximation in time. The resulting scheme can be expressed as

$$a_{\mathrm{P}} T_i^{n+1} + a_{\mathrm{E}} T_{i+1}^{n+1} + a_{\mathrm{W}} T_{i-1}^{n+1} = a_{\mathrm{P}}^{0} T_i^{n} + a_{\mathrm{P}}^{-1} T_i^{n-1}$$

What are the values of coefficients a_{P}, a_{W}, …. etc. for this scheme?

VII.8. Consider one-dimensional generic transport equation for scalar ϕ in the presence of uniform fluid velocity $u > 0$ in the positive x-direction

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = \frac{\Gamma}{\rho}\frac{\partial^2 \phi}{\partial x^2}.$$

a) Use the three-point backward difference formula derived in Module III for approximation of the time derivative and central difference scheme for spatial derivatives to derive an **implicit** time integration scheme.

b) If the three-point backward difference formula of part (a) is used as an upwind difference approximation of the convective term, what would be the form of the explicit time integration scheme based on **Euler's** method? What would be the stability properties of this scheme?

NOTE: For stability analysis of an explicit time integration scheme which can expressed as

$$\phi_i^{n+1} = \sum_i a_i \phi_i^n + \sum_j b_j \phi_j^{n-1}$$

use the criterion that for stability all a_i's and b_j's must be positive.