MODULE III

FINITE DIFFERENCE METHOD: ASSIGNMENT

III.1 Consider a continuous function $f(x)$, and a uniform grid with spacing Δx.

(a) Use the Taylor series expansion to obtain the finite difference approximations to the first and second order derivatives at the grid point x_i.

(b) Polynomial fitting method can be used to obtain the approximations to the first and second order derivatives at the grid point x_i. Derive these approximations using (i) quadratic, (ii) cubic and (iii) fourth order polynomials.

(c) What will be the form of the formulae derived in (b) for a boundary point (say, i =1)

III.2 Outline the steps involved in numerical simulation of a problem using finite difference method.

III.3 Consider the heat conduction in a slab bounded by planes $x = 0$ and $x = l$. The flux boundary condition is specified at $x = 0$. Use the polynomial fitting method on an uniform grid with spacing Δx to show that a second order accurate finite difference approximation for the first order derivative of the temperature at $x = 0$ would be given by

$$\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)_1 \approx \frac{4T_2 - T_3 - 3T_1}{2\Delta x}$$

III.4 Consider the viscous flow of air over a flat plate. To estimate the wall shear stress, τ_w, flow velocity, u, is measured at various points in a direction perpendicular to the plate (y-direction) at a given station. Let us use index 1 for a point on the bed at this station (i.e. $y_1 = 0$). The measured values of velocity at adjacent vertical nodes are

Node index	Node location, y (in mm)	Velocity, $u(y)$ in m/s
2	3	53.7
3	6	101.4
4	9	143.8

Calculation of the wall shear stress τ_w requires use of a one-sided difference formula. Derive the second and third order one-sided difference formulae using *polynomial fitting* for calculation of $(\mathrm{d}u/\mathrm{d}y)$ at node 1 ($y = 0$). Use these formulae as well as the first order forward difference scheme to compute the wall shear stress, τ_w. Assume air viscosity $\mu = 1.846\times10^{-5}$ N s/m^2.

III.5 Consider steady state heat conduction with constant heat generation $q_g = 5$ in a slab of constant conductivity $k = 1$ bounded by planes $x = 0$ and $x = 4$. Both ends of the slab are kept at zero temperature. Write down the governing equation and boundary conditions for the problem. Obtain an approximate numerical solution of the problem using the finite difference method (central difference scheme). Choose $\Delta x = 1$, and **use the TDMA.** For numerical solution, first do hand calculations and then write a computer program which can take different values of Δx. Compare results with analytical solution.

III.6 Velocity potential in an ideal flow is governed by Laplace equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

Use finite difference method to derive discretized equations for numerical solution of this problem. Write a simple computer code based on this finite difference problem and an iterative solver (say, SOR or PCG) for computation of potential in a simple rectangular domain. Also include subroutines which can compute velocity field and stream function.

III.7 Consider fully developed laminar viscous flow between two large parallel walls. Starting from Navier-Stokes equations, derive simplified governing equations for this problem. Derive discretized equations and write a computer code to solve this problem using finite difference method. Compare your results with analytical solution.

III.8 In a heat-treatment process, a very long slab of width 8 m has been heated to 400 °C. It is then cooled by bringing both of its sides in contact with a fluid bath at 0 °C. Assuming constant material properties, write down the governing equation, and initial and boundary conditions for the problem. Use the finite difference method (central difference scheme) for spatial discretization and two-level methods for time integration. Assume the diffusivity κ = 1 m^2/s. Choose Δx = 1 m and Δt = 0.25 s. Write a computer program to obtain the temperature distribution in the slab at time t = 0.25 s and t = 0.5 s using (a) forward Euler method, and (b) Crank-Nicolson method. Use TDMA for solution of linear system if required.

III.9 Consider the steady state heat conduction in a slab of width l = 0.5 m with heat generation. The left end of the slab (x = 0) is maintained at T = 373 K. The right end of the slab (x = 0.5 m) is being heated by a heater for which the heat flux is 1 kW/m^2. The heat generation in the slab is temperature dependent and is given by Q = (1273 – T) W/m^3. Thermal conductivity is constant at k = 1 W/(m-K).Write down the governing equation and boundary conditions for the problem. Use the finite difference method (central difference scheme) to obtain an approximate numerical solution of the problem. For the first order derivative, use forward or backward difference approximation of first order. Choose Δx = 0.1, and **use the TDMA.** Compare your results with analytical solution. For your reference, the formulas involved in the TDMA for solution of the system formed by the finite difference equations

$$A_W^i\phi_{i-1}+A_P^i\phi_i+A_E^i\phi_{i+1}=b_i$$

are:

$$A_P^i \leftarrow A_P^i-\frac{A_W^i A_E^{i-1}}{A_P^{i-1}}, \qquad b_i^*=b_i-\frac{A_W^i b_{i-1}^*}{A_P^{i-1}}, \qquad \phi_i=\frac{b_i^*-A_E^i\phi_{i+1}}{A_P^i}$$

III.10 Finite difference approximation of the three dimensional steady-state heat conduction equation $\nabla^2 T+\dot{q}=0$ using seven point computational molecule leads to a linear algebraic **Ax** = **b** where **A** is

(a) a dense square matrix.
(b) a tri-diagonal matrix.
(c) a sparse matrix.
(d) a seven-diagonal matrix.

SAMPLE SOLUTION (PROBLEM III.9)

Governing equation for the steady state heat conduction with constant heat generation is the slab is

$$k\frac{\mathrm{d}^2T}{\mathrm{d}x^2}+Q=0 \tag{1}$$

Given: $Q = 1273 - T$. Thus, eqn. (1) becomes

$$k\frac{\mathrm{d}^2T}{\mathrm{d}x^2}-T=-1273 \tag{2}$$

Left end of the slab is maintained at constant temperature; hence boundary condition at this end is given by

$$T(0)=373 \tag{3}$$

At the right end, heat influx is specified. Thus, boundary condition at this end is

$$k\frac{\mathrm{d}T}{\mathrm{d}x}=1000 \tag{4}$$

Using central difference scheme, the discretized form of (2) at an internal node $x = x_i$ can be expressed as

$$\frac{T_{i+1}+T_{i-1}-2T_i}{\Delta x^2}-T_i=-1273 \tag{5}$$

Using given values of $k = 1$ and $\Delta x = 0.1$, the preceding equation simplifies to

$$-T_{i-1}+2.01T_i-T_{i+1}=12.73, \quad i=2,3,4,5 \tag{6}$$

Hence, the coefficients in the standard equation $A_W^iT_{i-1}+A_P^iT_i+A_E^iT_{i+1}=b_i$ are:

$$A_W^i=-1,\ A_P^i=2.01, \quad A_E^i=-1, \quad b_i=12.73, \quad i=2,3,4,5 \tag{7}$$

Temperature boundary condition (3) at the first node implies

$$T_1=373,\ \text{i.e.}\ A_W^1=0,\ A_P^1=1, \quad A_E^1=0, \quad b_1=373 \tag{8}$$

Discretization of the flux boundary (4) requires use of backward difference, which yields

$$T_6-T_5=100,\ \text{i.e.}\ A_W^6=-1,\ A_P^6=1, \quad A_E^6=0, \quad b_6=100 \tag{9}$$

Numerical calculations using TDMA are given in the following table:

i	A_W^i	A_P^i	A_E^i	b_i	$A_P^i \leftarrow A_P^i - \frac{A_W^iA_E^{i-1}}{A_P^{i-1}}$	$b_i^* = b_i - \frac{A_W^ib_{i-1}^*}{A_P^{i-1}}$	$T_i = \frac{b_i^* - A_E^iT_{i+1}}{A_P^i}$	T_{ex}
1	0	1.00	0	373	1	373	373.0000	373.00
2	-1	2.01	-1	12.73	2.01	385.73	497.2893	498.95
3	-1	2.01	-1	12.73	1.512487	204.6355	613.8216	617.18
4	-1	2.01	-1	12.73	1.34883754	148.02734	723.7617	728.85
5	-1	2.01	-1	12.73	1.26862087	122.47438	828.2096	835.08
6	-1	1.00	0	100	0.21174243	196.54136	928.2096	936.92