

Axial Deformations

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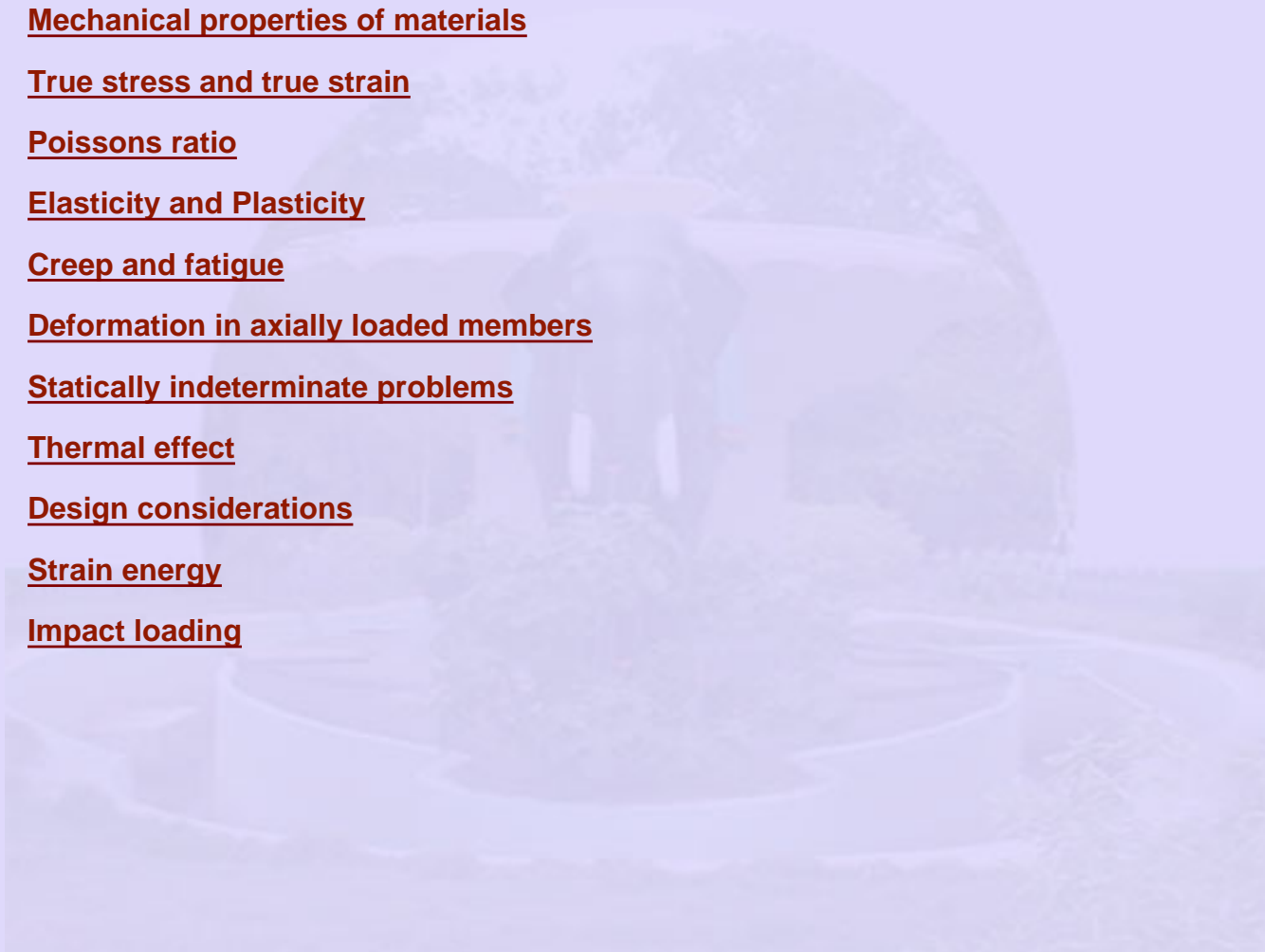
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1.1 Introduction

An important aspect of the analysis and design of structures relates to the deformations caused by the loads applied to a structure. Clearly it is important to avoid deformations so large that they may prevent the structure from fulfilling the purpose for which it is intended. But the analysis of deformations may also help us in the determination of stresses. It is not always possible to determine the forces in the members of a structure by applying only the principle of statics. This is because statics is based on the assumption of undeformable, rigid structures. By considering engineering structures as deformable and analyzing the deformations in their various members, it will be possible to compute forces which are statically indeterminate. Also the distribution of stresses in a given member is indeterminate, even when the force in that member is known. To determine the actual distribution of stresses within a member, it is necessary to analyze the deformations which take place in that member. This chapter deals with the deformations of a structural member such as a rod, bar or a plate under *axial loading*.



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1.2 Free body diagram - Revisited

The first step towards solving an engineering problem is drawing the free body diagram of the element/structure considered.

Removing an existing force or including a wrong force on the free body will badly affect the equilibrium conditions, and hence, the analysis.

In view of this, some important points in drawing the free body diagram are discussed below.

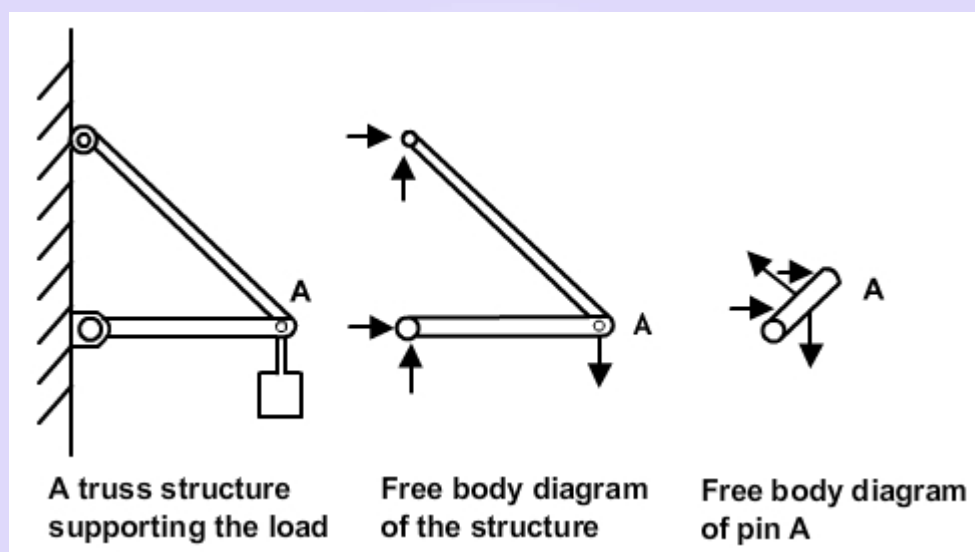


Figure 1.1

At the beginning, a clear decision is to be made by the analyst on the choice of the body to be considered for free body diagram.

Then that body is detached from all of its surrounding members including ground and only their forces on the free body are represented.

The weight of the body and other external body forces like centrifugal, inertia, etc., should also be included in the diagram and they are assumed to act at the centre of gravity of the body.

When a structure involving many elements is considered for free body diagram, the forces acting in between the elements should not be brought into the diagram.

The known forces acting on the body should be represented with proper magnitude and direction.

If the direction of unknown forces like reactions can be decided, they should be indicated clearly in the diagram.

After completing free body diagram, equilibrium equations from statics in terms of forces and moments are applied and solved for the unknowns.

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1.3 Normal, shear and bearing stress

1.3.1 Normal Stress:

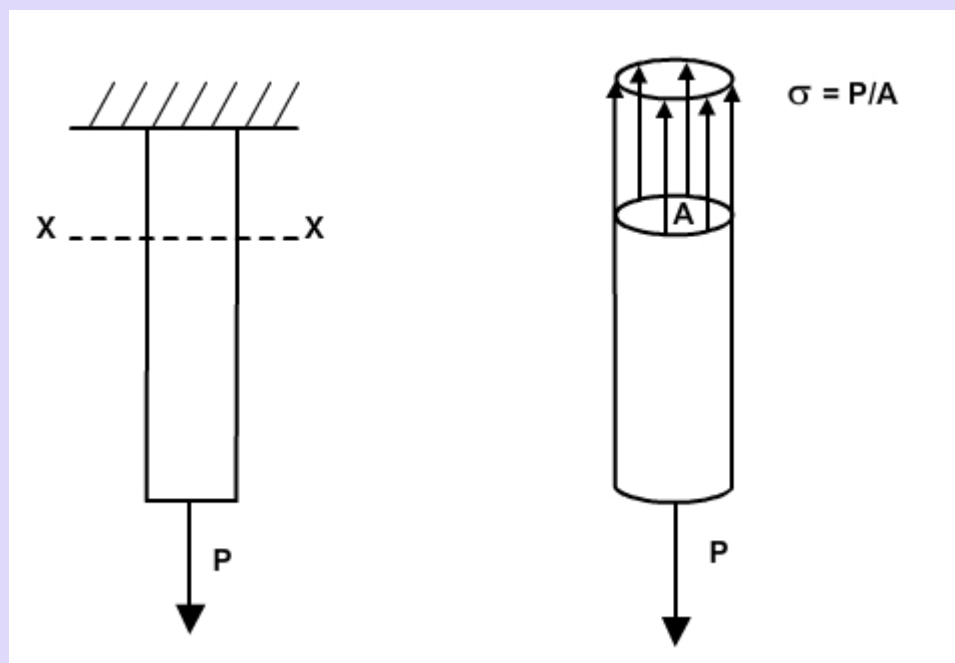


Figure 1.2

When a structural member is under load, predicting its ability to withstand that load is not possible merely from the reaction force in the member.

It depends upon the internal force, cross sectional area of the element and its material properties.

Thus, a quantity that gives the ratio of the internal force to the cross sectional area will define the ability of the material in with standing the loads in a better way.

That quantity, i.e., the intensity of force distributed over the given area or simply the force per unit area is called the *stress*.

$$\sigma = \frac{P}{A} \quad 1.1$$

In SI units, force is expressed in newtons (N) and area in square meters. Consequently, the stress has units of newtons per square meter (N/m^2) or Pascals (Pa).

In figure 1.2, the stresses are acting normal to the section XX that is perpendicular to the axis of the bar. These stresses are called *normal stresses*.

The stress defined in equation 1.1 is obtained by dividing the force by the cross sectional area and hence it represents the average value of the stress over the entire cross section.

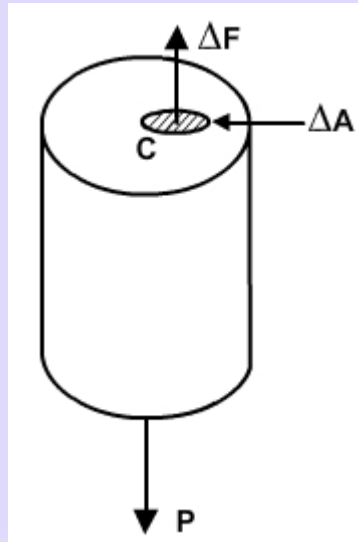


Figure 1.3

Consider a small area ΔA on the cross section with the force acting on it ΔF as shown in figure 1.3. Let the area contain a point C.

Now, the stress at the point C can be defined as,

$$\sigma = \lim_{\Delta A \rightarrow 0} \frac{\Delta F}{\Delta A} \quad 1.2$$

The average stress values obtained using equation 1.1 and the stress value at a point from equation 1.2 may not be the same for all cross sections and for all loading conditions.

1.3.2 Saint - Venant's Principle:

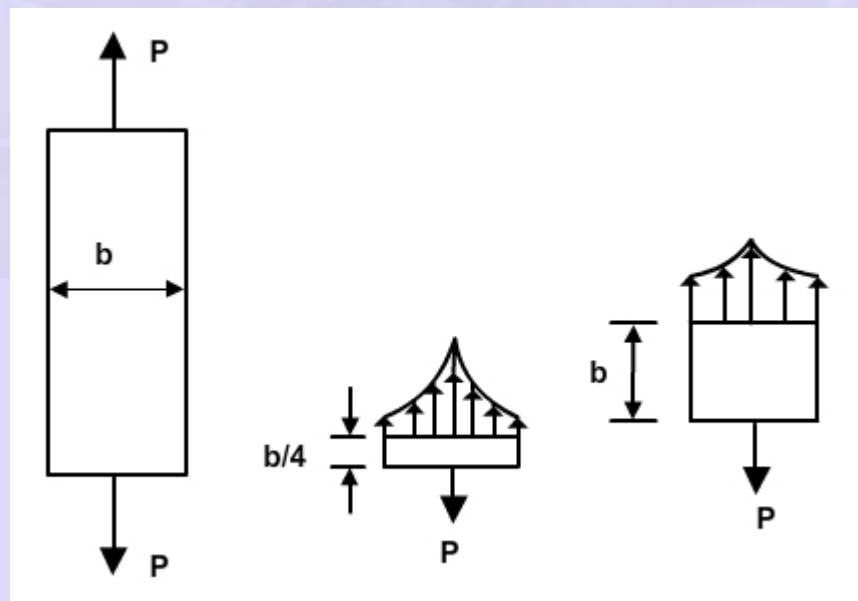


Figure 1.4

Consider a slender bar with point loads at its ends as shown in figure 1.4.

The normal stress distribution across sections located at distances $b/4$ and b from one end of the bar is represented in the figure.

It is found from figure 1.4 that the stress varies appreciably across the cross section in the immediate vicinity of the application of loads.

The points very near the application of the loads experience a larger stress value whereas, the points far away from it on the same section has lower stress value.

The variation of stress across the cross section is negligible when the section considered is far away, about equal to the width of the bar, from the application of point loads.

Thus, except in the immediate vicinity of the points where the load is applied, the stress distribution may be assumed to be uniform and is independent of the mode of application of loads. This principle is called *Saint-Venant's principle*.

1.3.3 Shear Stress:

The stresses acting perpendicular to the surfaces considered are normal stresses and were discussed in the preceding section.

Now consider a bolted connection in which two plates are connected by a bolt with cross section A as shown in figure 1.5.

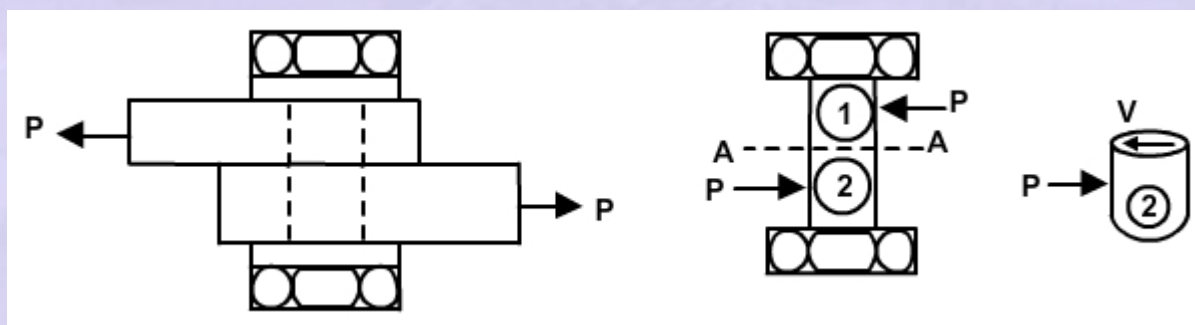


Figure 1.5

The tensile loads applied on the plates will tend to shear the bolt at the section AA.

Hence, it can be easily concluded from the free body diagram of the bolt that the internal resistance force V must act in the plane of the section AA and it should be equal to the external load P .

These internal forces are called shear forces and when they are divided by the corresponding section area, we obtain the *shear stress* on that section.

$$\tau = \frac{V}{A} \quad 1.3$$

Equation 1.3 defines the average value of the shear stress on the cross section and the distribution of them over the area is not uniform.

In general, the shear stress is found to be maximum at the centre and zero at certain locations on the edge. This will be dealt in detail in shear stresses in beams (module 6).

In figure 1.5, the bolt experiences shear stresses on a single plane in its body and hence it is said to be under single shear.

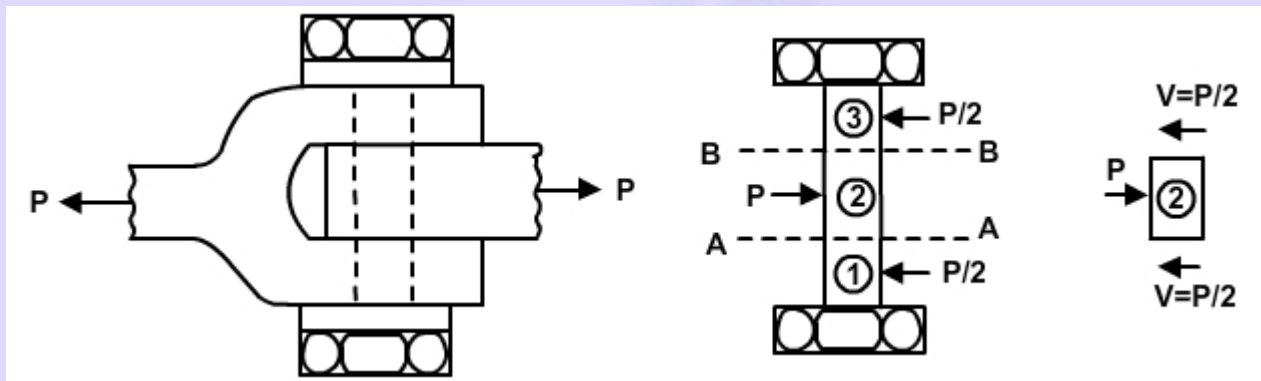


Figure 1.6

In figure 1.6, the bolt experiences shear on two sections AA and BB. Hence, the bolt is said to be under double shear and the shear stress on each section is

$$\tau = \frac{V}{A} = \frac{P}{2A} \quad 1.4$$

Assuming that the same bolt is used in the assembly as shown in figure 1.5 and 1.6 and the same load P is applied on the plates, we can conclude that the shear stress is reduced by half in double shear when compared to a single shear.

Shear stresses are generally found in bolts, pins and rivets that are used to connect various structural members and machine components.

1.3.4 Bearing Stress:

In the bolted connection in figure 1.5, a highly irregular pressure gets developed on the contact surface between the bolt and the plates.

The average intensity of this pressure can be found out by dividing the load P by the projected area of the contact surface. This is referred to as the *bearing stress*.

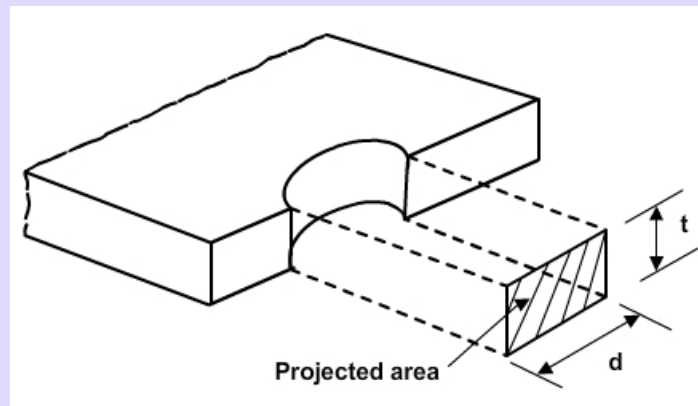


Figure 1.7

The projected area of the contact surface is calculated as the product of the diameter of the bolt and the thickness of the plate.

Bearing stress,

$$\sigma_b = \frac{P}{A} = \frac{P}{t \times d} \quad 1.5$$

Example 1:

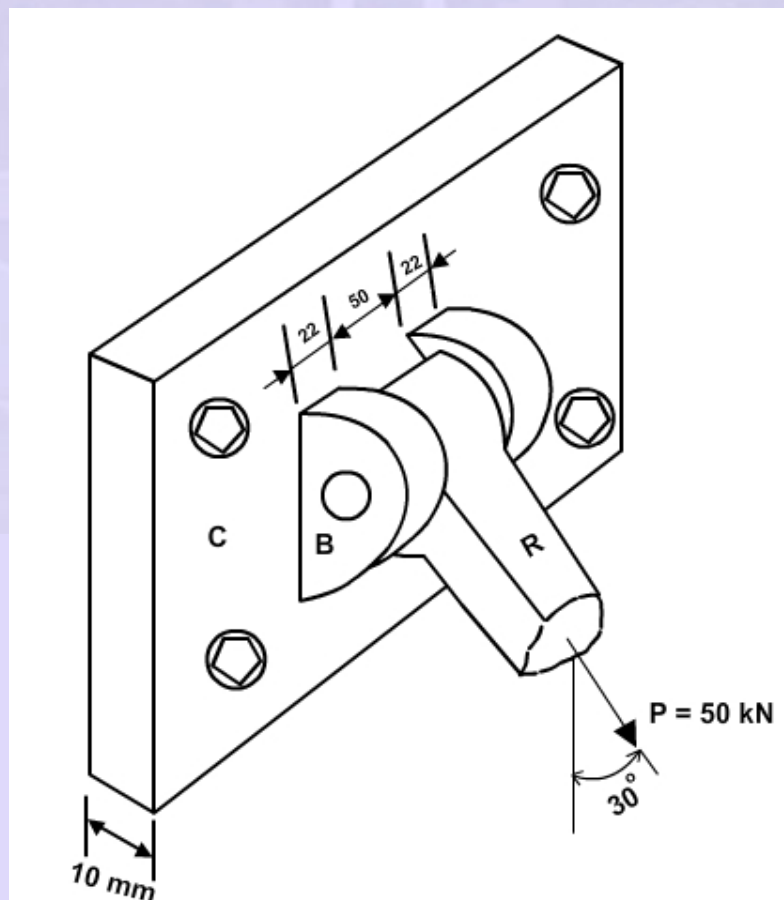


Figure 1.8

A rod R is used to hold a sign board with an axial load 50 kN as shown in figure 1.8. The end of the rod is 50 mm wide and has a circular hole for fixing the pin which is 20 mm diameter. The load from the rod R is transferred to the base plate C through a bracket B that is 22mm wide and has a circular hole for the pin. The base plate is 10 mm thick and it is connected to the bracket by welding. The base plate C is fixed on to a structure by four bolts of each 12 mm diameter. Find the shear stress and bearing stress in the pin and in the bolts.

Solution:

$$\text{Shear stress in the pin} = \tau_{\text{pin}} = \frac{V}{A} = \frac{P}{2A} = \frac{(50 \times 10^3)/2}{\pi(0.02)^2/4} = 79.6 \text{ MPa}$$

$$\text{Force acting on the base plate} = P \cos \theta = 50 \cos 30^\circ = 43.3 \text{ kN}$$

$$\text{Shear stress in the bolt, } \tau_{\text{bolt}} = \frac{P}{4A} = \frac{(43.3 \times 10^3)/4}{\pi(0.012)^2/4}$$

$$= 95.7 \text{ MPa}$$

$$\text{Bearing stress between pin and rod, } \sigma_b = \frac{P}{b \times d} = \frac{(50 \times 10^3)}{(0.05) \times (0.02)}$$

$$= 50 \text{ MPa}$$

$$\text{Bearing stress between pin and bracket} = \sigma_b = \frac{P/2}{b \times d} = \frac{(50 \times 10^3)/2}{(0.022) \times (0.02)}$$

$$= 56.8 \text{ MPa}$$

$$\text{Bearing stress between plate and bolts} = \sigma_b = \frac{P/4}{t \times d} = \frac{(43.3 \times 10^3)/4}{(0.01) \times (0.012)}$$

$$= 90.2 \text{ MPa}$$

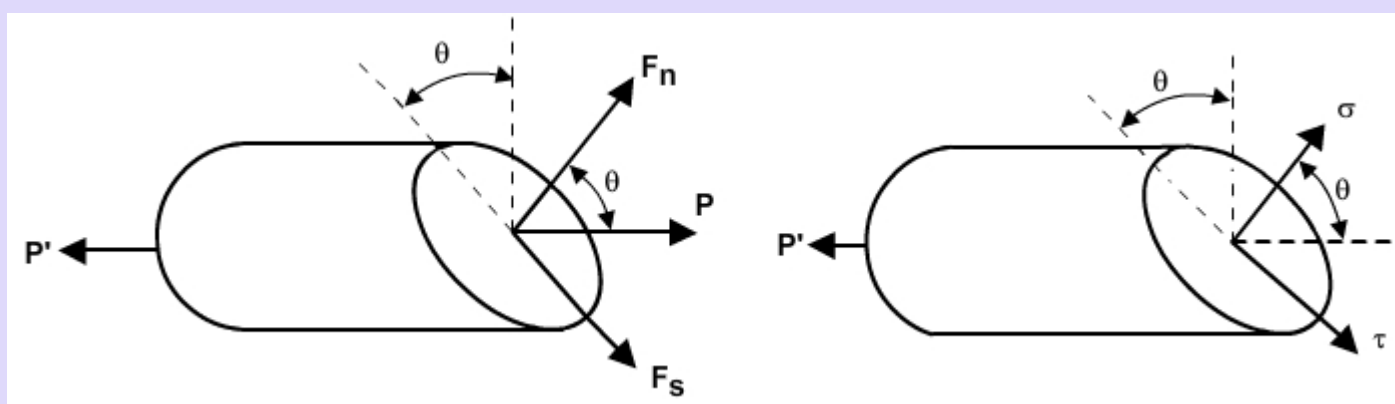
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1.4 Stress on inclined planes under axial loading:

When a body is under an axial load, the plane normal to the axis contains only the normal stress as discussed in section 1.3.1.

However, if we consider an oblique plane that forms an angle θ with normal plane, it consists shear stress in addition to normal stress.

Consider such an oblique plane in a bar. The resultant force P acting on that plane will



keep the bar in equilibrium against the external load P' as shown in figure 1.9.

Figure 1.9

The resultant force P on the oblique plane can be resolved into two components F_n and F_s that are acting normal and tangent to that plane, respectively.

If A is the area of cross section of the bar, $A/\cos\theta$ is the area of the oblique plane. Normal and shear stresses acting on that plane can be obtained as follows.

$$F_n = P \cos \theta$$

$$F_s = -P \sin \theta \text{ (Assuming shear causing clockwise rotation negative).}$$

$$\sigma = \frac{P \cos \theta}{A / \cos \theta} = \frac{P}{A} \cos^2 \theta \quad 1.6$$

$$\tau = -\frac{P \sin \theta}{A / \cos \theta} = -\frac{P}{A} \sin \theta \cos \theta \quad 1.7$$

Equations 1.6 and 1.7 define the normal and shear stress values on an inclined plane that makes an angle θ with the vertical plane on which the axial load acts.

From above equations, it is understandable that the normal stress reaches its maximum when $\theta = 0^\circ$ and becomes zero when $\theta = 90^\circ$.

But, the shear stress assumes zero value at $\theta = 0^\circ$ and $\theta = 90^\circ$ and reaches its maximum when $\theta = 45^\circ$.

The magnitude of maximum shear stress occurring at $\theta = 45^\circ$ plane is half of the maximum normal stress that occurs at $\theta = 0^\circ$ for a material under a uniaxial loading.

$$\tau_{\max} = \frac{P}{2A} = \frac{\sigma_{\max}}{2} \quad 1.8$$

Now consider a cubic element A in the rod which is represented in two dimension as shown in figure 1.10 such that one of its sides makes an angle θ with the vertical plane.

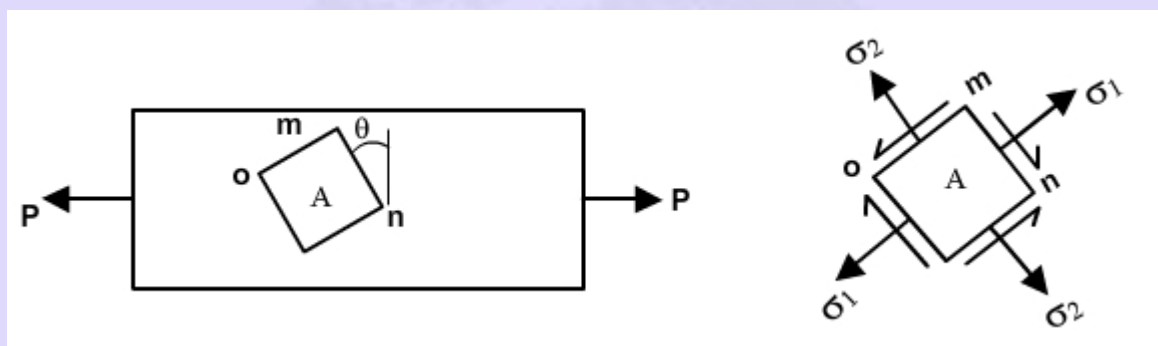


Figure 1.10

To determine the stresses acting on the plane mn , equations 1.6 and 1.7 are used as such and to know the stresses on plane om , θ is replaced by $\theta + 90^\circ$.

Maximum shear stress occurs on both om and mn planes with equal magnitude and opposite signs, when mn forms 45° angle with vertical plane.

Example 2:

A prismatic bar of sides 40 mm x 30 mm is axially loaded with a compressive force of 80 kN. Determine the stresses acting on an element which makes 30° inclination with the vertical plane. Also find the maximum shear stress value in the bar.

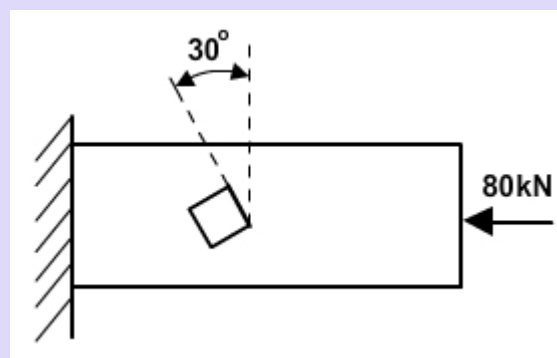


Figure 1.11

Solution:

Area of the cross section, $A = 40 \times 30 \times 10^{-6} = 1.2 \times 10^{-3} \text{ m}^2$

Normal stress on 30° inclined plane, $\sigma = \frac{P}{A} \cos^2 \theta$

$$= \frac{-80 \times 10^3}{1.2 \times 10^{-3}} \times \cos^2 30^\circ = -50 \text{ MPa}$$

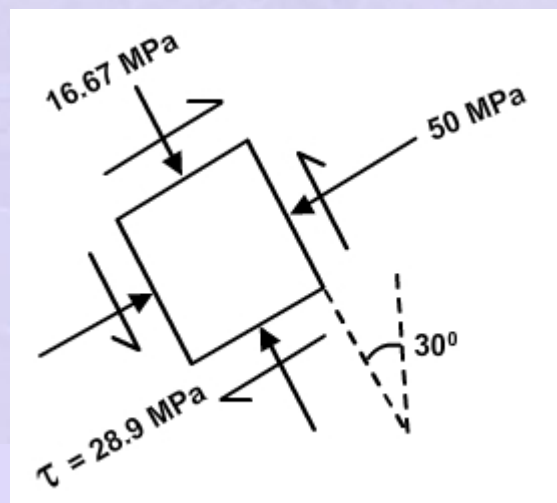
Shear stress on 30° plane, $\tau = \frac{-P}{A} \sin \theta \cos \theta = \frac{80 \times 10^3}{1.2 \times 10^{-3}} \times \sin 30^\circ \times \cos 30^\circ$

$$= 28.9 \text{ MPa [Counter clockwise]}$$

Normal stress on 120° plane, $\sigma = \frac{-80 \times 10^3}{1.2 \times 10^{-3}} \cos^2 120^\circ = -16.67 \text{ MPa}$

Shear stress on 120° plane, $\tau = \frac{80 \times 10^3}{1.2 \times 10^{-3}} \times \sin 120^\circ \times \cos 120^\circ = -28.9 \text{ MPa [Clock wise]}$

Maximum shear stress in the bar, $\tau_{\max} = \frac{P}{2A} = \frac{80 \times 10^3}{2 \times 1.2 \times 10^{-3}}$
 $= \pm 33.3 \text{ MPa}$



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1.5 Strain

The structural member and machine components undergo deformation as they are brought under loads.

To ensure that the deformation is within the permissible limits and do not affect the performance of the members, a detailed study on the deformation assumes significance.

A quantity called *strain* defines the deformation of the members and structures in a better way than the deformation itself and is an indication on the state of the material.

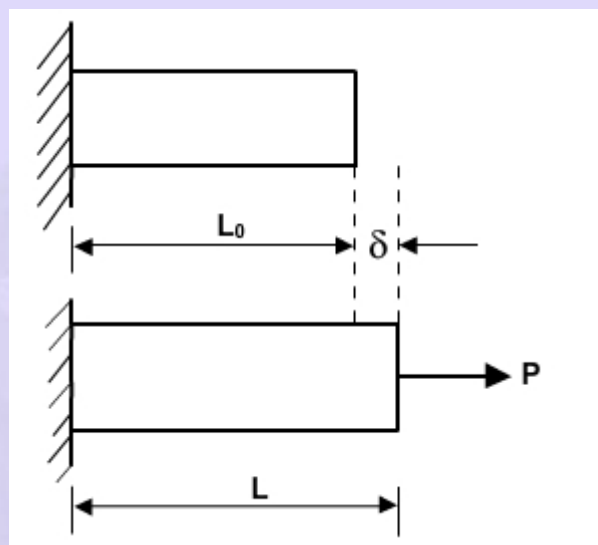


Figure 1.12

Consider a rod of uniform cross section with initial length L_0 as shown in figure 1.12. Application of a tensile load P at one end of the rod results in elongation of the rod by δ . After elongation, the length of the rod is L . As the cross section of the rod is uniform, it is appropriate to assume that the elongation is uniform throughout the volume of the rod. If the tensile load is replaced by a compressive load, then the deformation of the rod will be a contraction. The deformation per unit length of the rod along its axis is defined as the *normal strain*. It is denoted by ε

$$\varepsilon = \frac{\delta}{L} = \frac{L - L_0}{L} \quad 1.9$$

Though the strain is a dimensionless quantity, units are often given in mm/mm, $\mu\text{m}/\text{m}$.

Example 3:

A circular hollow tube made of steel is used to support a compressive load of 500kN. The inner and outer diameters of the tube are 90mm and 130mm respectively and its length is 1000mm. Due to compressive load, the contraction of the rod is 0.5mm. Determine the compressive stress and strain in the post.

Solution

Force, $P = -500 \times 10^3 \text{ N}$ (compressive)

Area of the tube, $A = \frac{\pi}{4} \left[(0.13)^2 - (0.09)^2 \right] = 6.912 \times 10^{-3} \text{ m}^2$

Stress, $\sigma = \frac{P}{A} = \frac{-500 \times 10^3}{6.912 \times 10^{-3}} = -72.3 \text{ MPa}$ (compressive)

Strain, $\epsilon = \frac{\delta}{L_0} = \frac{-0.5}{1000} = -5 \times 10^{-4}$ (compressive)



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1.6 Mechanical properties of materials

A tensile test is generally conducted on a standard specimen to obtain the relationship between the stress and the strain which is an important characteristic of the material.

In the test, the uniaxial load is applied to the specimen and increased gradually. The corresponding deformations are recorded throughout the loading.

Stress-strain diagrams of materials vary widely depending upon whether the material is ductile or brittle in nature.

If the material undergoes a large deformation before failure, it is referred to as ductile material or else brittle material.

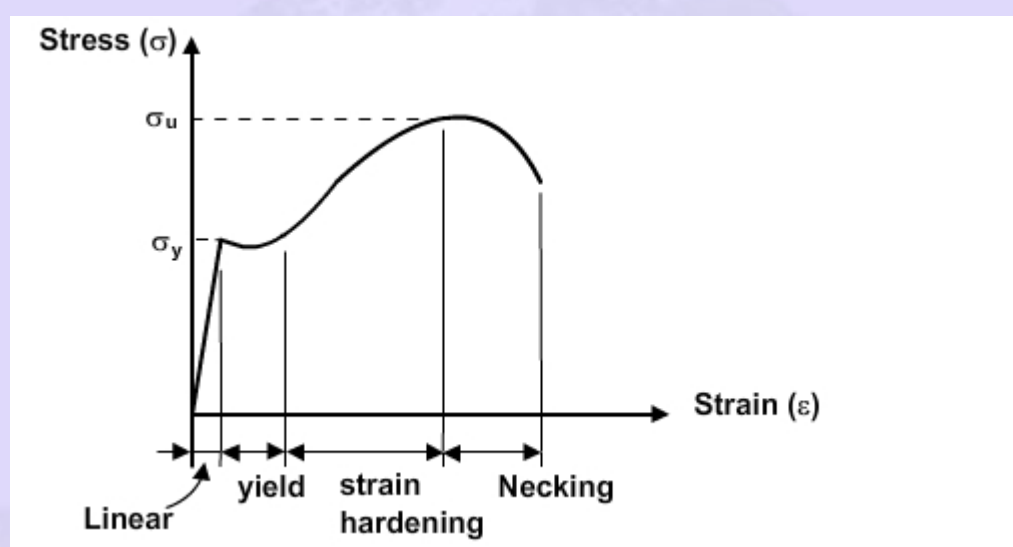


Figure 1.13

In figure 1.13, the stress-strain diagram of a structural steel, which is a ductile material, is given.

Initial part of the loading indicates a linear relationship between stress and strain, and the deformation is completely recoverable in this region for both ductile and brittle materials.

This linear relationship, i.e., stress is directly proportional to strain, is popularly known as *Hooke's law*.

$$\sigma = E\varepsilon \quad 1.10$$

The co-efficient E is called the *modulus of elasticity* or *Young's modulus*.

Most of the engineering structures are designed to function within their linear elastic region only.

After the stress reaches a critical value, the deformation becomes irrecoverable. The corresponding stress is called the *yield stress* or yield strength of the material beyond which the material is said to start yielding.

In some of the ductile materials like low carbon steels, as the material reaches the yield strength it starts yielding continuously even though there is no increment in external load/stress. This flat curve in stress strain diagram is referred as perfectly plastic region.

The load required to yield the material beyond its yield strength increases appreciably and this is referred to strain hardening of the material.

In other ductile materials like aluminum alloys, the strain hardening occurs immediately after the linear elastic region without perfectly elastic region.

After the stress in the specimen reaches a maximum value, called *ultimate strength*, upon further stretching, the diameter of the specimen starts decreasing fast due to local instability and this phenomenon is called *necking*.

The load required for further elongation of the material in the necking region decreases with decrease in diameter and the stress value at which the material fails is called the *breaking strength*.

In case of brittle materials like cast iron and concrete, the material experiences smaller deformation before rupture and there is no necking.

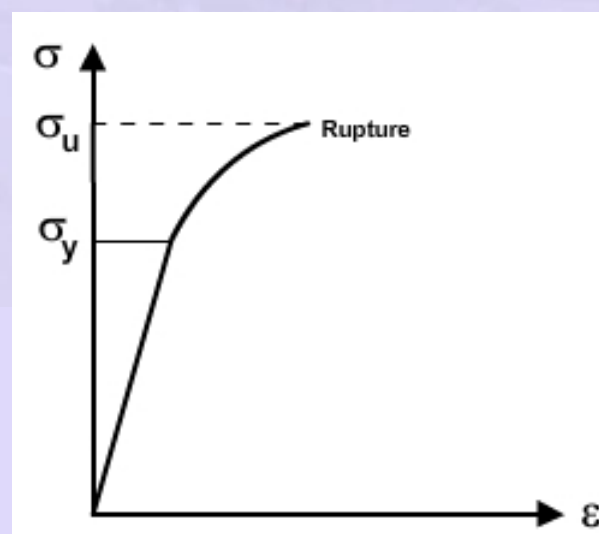


Figure 1.14

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1.7 True stress and true strain

In drawing the stress-strain diagram as shown in figure 1.13, the stress was calculated by dividing the load P by the initial cross section of the specimen.

But it is clear that as the specimen elongates its diameter decreases and the decrease in cross section is apparent during necking phase.

Hence, the actual stress which is obtained by dividing the load by the actual cross sectional area in the deformed specimen is different from that of the engineering stress that is obtained using undeformed cross sectional area as in equation 1.1

True stress or actual stress,

$$\sigma_{\text{act}} = \frac{P}{A_{\text{act}}} \quad 1.11$$

Though the difference between the true stress and the engineering stress is negligible for smaller loads, the former is always higher than the latter for larger loads.

Similarly, if the initial length of the specimen is used to calculate the strain, it is called engineering strain as obtained in equation 1.9

But some engineering applications like metal forming process involve large deformations and they require actual or *true strains* that are obtained using the successive recorded lengths to calculate the strain.

$$\text{True strain} = \int_{L_0}^L \frac{dL}{L} = \ln \left(\frac{L}{L_0} \right) \quad 1.12$$

True strain is also called as actual strain or natural strain and it plays an important role in theories of viscosity.

The difference in using engineering stress-strain and the true stress-strain is noticeable after the proportional limit is crossed as shown in figure 1.15.

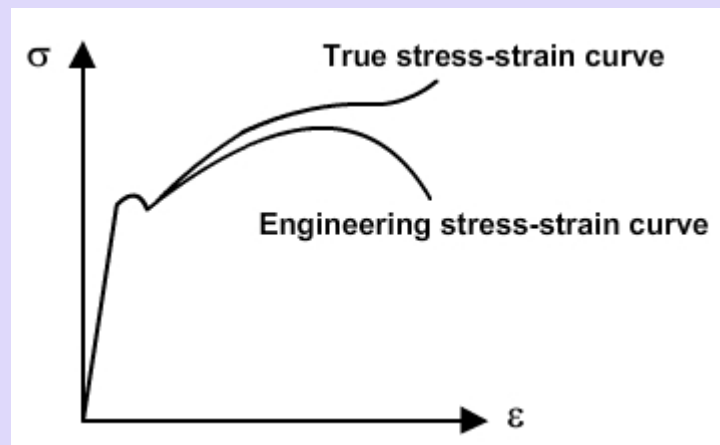


Figure 1.15

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1.8 Poissons ratio

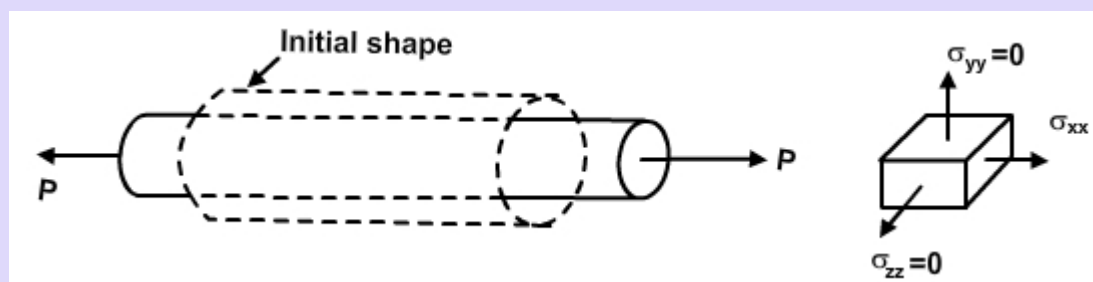


Figure 1.16

Consider a rod under an axial tensile load P as shown in figure 1.6 such that the material is within the elastic limit. The normal stress on x plane is $\sigma_{xx} = \frac{P}{A}$ and the associated longitudinal strain in the x direction can be found out from $\epsilon_x = \frac{\sigma_{xx}}{E}$. As the material elongates in the x direction due to the load P , it also contracts in the other two mutually perpendicular directions, i.e., y and z directions.

Hence, despite the absence of normal stresses in y and z directions, strains do exist in those directions and they are called lateral strains.

The ratio between the lateral strain and the axial/longitudinal strain for a given material is always a constant within the elastic limit and this constant is referred to as *Poisson's ratio*. It is denoted by ν .

$$\nu = - \frac{\text{lateral strain}}{\text{axial strain}} \quad 1.13$$

Since the axial and lateral strains are opposite in sign, a negative sign is introduced in equation 1.13 to make ν positive.

Using equation 1.13, the lateral strain in the material can be obtained by

$$\epsilon_y = \epsilon_z = -\nu\epsilon_x = -\nu \frac{\sigma_{xx}}{E} \quad 1.14$$

Poisson's ratio can be as low as 0.1 for concrete and as high as 0.5 for rubber.

In general, it varies from 0.25 to 0.35 and for steel it is about 0.3.

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1.9 Elasticity and Plasticity

If the strain disappears completely after removal of the load, then the material is said to be in elastic region.

The stress-strain relationship in elastic region need not be linear and can be non-linear as in rubber like materials.

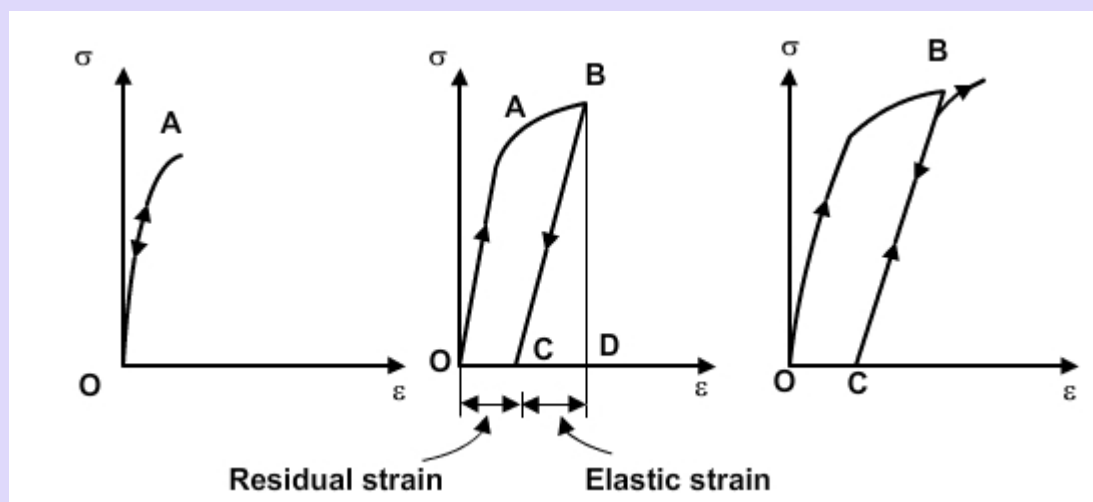


Figure 1.17

The maximum stress value below which the strain is fully recoverable is called the *elastic limit*. It is represented by point A in figure 1.17.

When the stress in the material exceeds the elastic limit, the material enters into plastic phase where the strain can no longer be completely removed.

To ascertain that the material has reached the plastic region, after each load increment, it is unloaded and checked for residual strain.

Presence of residual strain is the indication that the material has entered into plastic phase.

If the material has crossed elastic limit, during unloading it follows a path that is parallel to the initial elastic loading path with the same proportionality constant E .

The strain present in the material after unloading is called the residual strain or plastic strain and the strain disappears during unloading is termed as recoverable or elastic strain.

They are represented by OC and CD , respectively in figure.1.17.

If the material is reloaded from point C , it will follow the previous unloading path and line CB becomes its new elastic region with elastic limit defined by point B .

Though the new elastic region CB resembles that of the initial elastic region OA, the internal structure of the material in the new state has changed.

The change in the microstructure of the material is clear from the fact that the ductility of the material has come down due to strain hardening.

When the material is reloaded, it follows the same path as that of a virgin material and fails on reaching the ultimate strength which remains unaltered due to the intermediate loading and unloading process.

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1.10 Creep and fatigue

In the preceding section, it was discussed that the plastic deformation of a material increases with increasing load once the stress in the material exceeds the elastic limit.

However, the materials undergo additional plastic deformation with time even though the load on the material is unaltered.

Consider a bar under a constant axial tensile load as shown in figure 1.18.

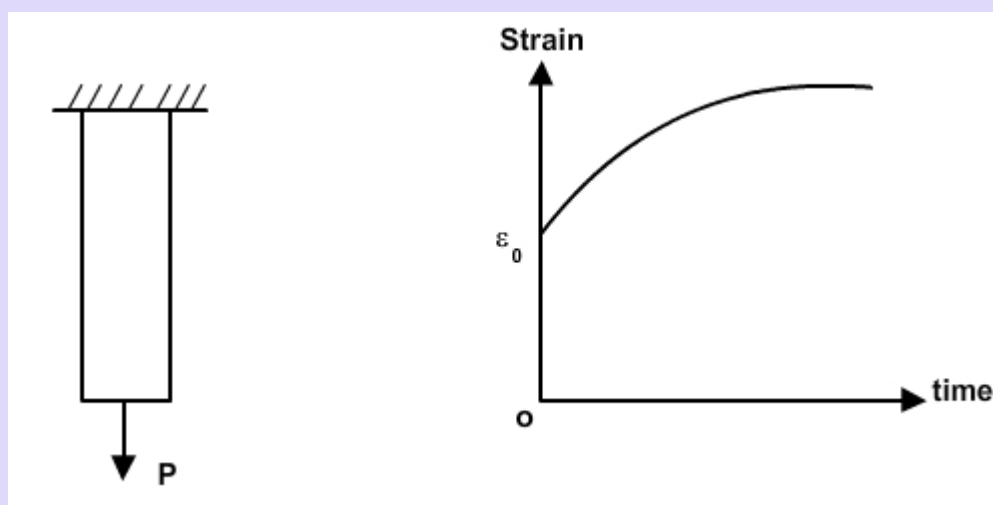


Figure 1.18

As soon as the material is loaded beyond its elastic limit, it undergoes an instant plastic deformation ϵ_0 at time $t = 0$.

Though the material is not brought under additional loads, it experiences further plastic deformation with time as shown in the graph in figure 1.18.

This phenomenon is called *creep*.

Creep at high temperature is of more concern and it plays an important role in the design of engines, turbines, furnaces, etc.

However materials like concrete, steel and wood experience creep slightly even at normal room temperature that is negligible.

Analogous to creep, the load required to keep the material under constant strain decreases with time and this phenomenon is referred to as *stress relaxation*.

It was concluded in section 1.9 that the specimen will not fail when the stress in the material is within the elastic limit.

This holds true only for static loading conditions and if the applied load fluctuates or reverses then the material will fail far below its yield strength.

This phenomenon is known as *fatigue*.

Designs involving fluctuating loads like traffic in bridges, and reversing loads like automobile axles require fatigue analysis.

Fatigue failure is initiated by a minute crack that develops at a high stress point which may be an imperfection in the material or a surface scratch.

The crack enlarges and propagates through the material due to successive loadings until the material fails as the undamaged portion of the material is insufficient to withstand the load.

Hence, a polished surface shaft can take more number of cycles than a shaft with rough or corroded surface.

The number of cycles that can be taken up by a material before it fractures can be found out by conducting experiments on material specimens.

The obtained results are plotted as $\sigma - n$ curves as given in figure 1.19, which indicates the number of cycles that can be safely completed by the material under a given maximum stress.

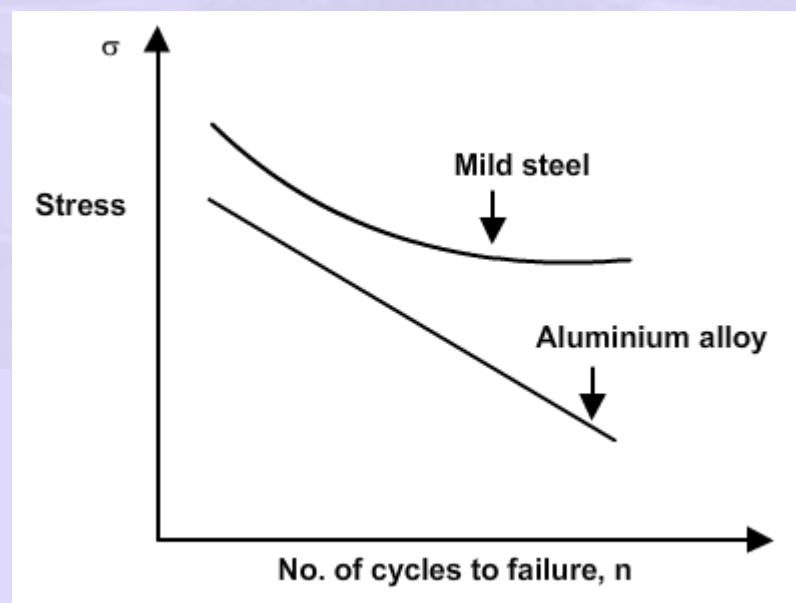


Figure 1.19

It is learnt from the graph that the number of cycles to failure increases with decrease in magnitude of stress.

For steels, if the magnitude of stress is reduced to a particular value, it can undergo an infinitely large number of cycles without fatigue failure and the corresponding stress is known as *endurance limit* or *fatigue limit*.

On the other hand, for non-ferrous metals like aluminum alloys there is no endurance limit, and hence, the maximum stress decreases continuously with increase in number of cycles. In such cases, the fatigue limit of the material is taken as the stress value that will allow an arbitrarily taken number of cycles, say 10^8 cycles.

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1.11 Deformation in axially loaded members

Consider the rod of uniform cross section under tensile load P along its axis as shown in figure 1.12.

Let that the initial length of the rod be L and the deflection due to load be δ . Using equations 1.9 and 1.10,

$$\begin{aligned}\frac{\delta}{L} &= \varepsilon = \frac{\sigma}{E} = \frac{P}{AE} \\ \delta &= \frac{PL}{AE}\end{aligned}\quad 1.15$$

Equation 1.15 is obtained under the assumption that the material is homogeneous and has a uniform cross section.

Now, consider another rod of varying cross section with the same axial load P as shown in figure 1.20.

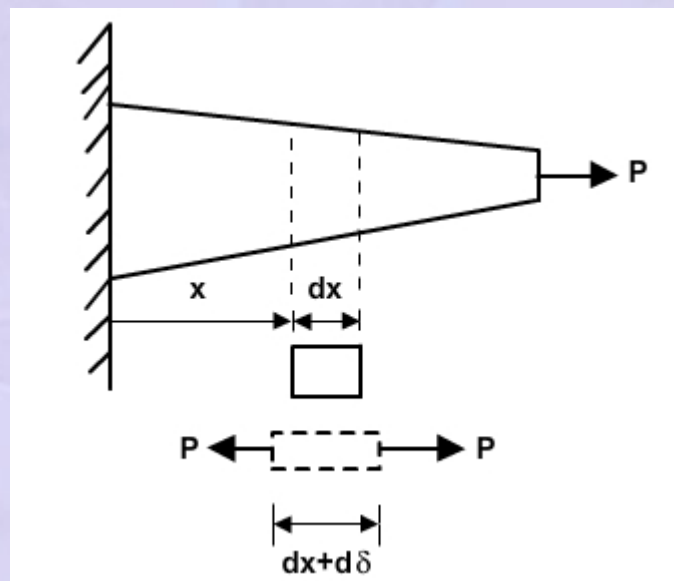


Figure 1.20

Let us take an infinitesimal element of length dx in the rod that undergoes a deflection

$d\delta$ due to load P . The strain in the element is $\varepsilon = \frac{d\delta}{dx}$ and $d\delta = \varepsilon dx$

The deflection of total length of the rod can be obtained by integrating above equation, $\delta = \int \varepsilon dx$

$$\delta = \int_0^L \frac{P dx}{EA(x)} \quad 1.16$$

As the cross sectional area of the rod keeps varying, it is expressed as a function of its length.

If the load is also varying along the length like the weight of the material, it should also be expressed as a function of distance, i.e., $P(x)$ in equation 1.16.

Also, if the structure consists of several components of different materials, then the deflection of each component is determined and summed up to get the total deflection of the structure.

When the cross section of the components and the axial loads on them are not varying along length, the total deflection of the structure can be determined easily by,

$$\delta = \sum_{i=1}^n \frac{P_i L_i}{A_i E_i} \quad 1.17$$

Example 4:

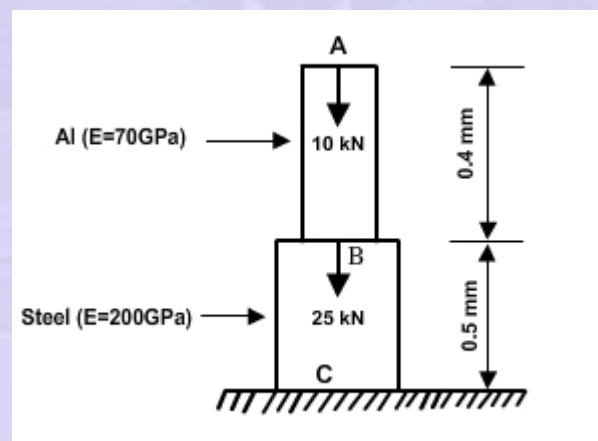
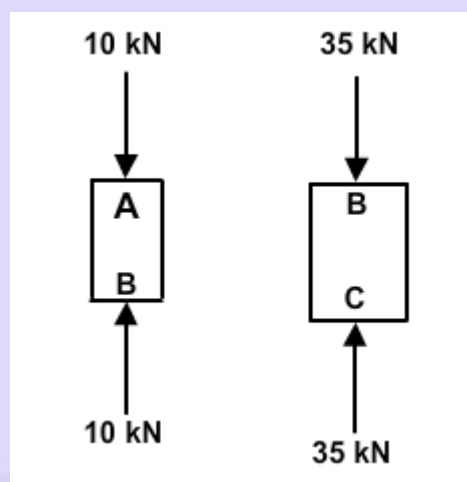


Figure 1.21

Consider a rod ABC with aluminum part AB and steel part BC having diameters 25mm and 50 mm respectively as shown figure 1.21. Determine the deflections of points A and B.

Solution:

$$\text{Deflection of part AB} = \frac{P_{AB}L_{AB}}{A_{AB}E_{AB}} = -\frac{10 \times 10^3 \times 400}{\pi \times (0.025)^2 / 4 \times 70 \times 10^9}$$

$$= -0.1164 \text{ mm}$$

$$\text{Deflection of part BC} = \frac{P_{BC}L_{BC}}{A_{BC}E_{BC}} = -\frac{35 \times 10^3 \times 500}{\pi \times (0.05)^2 / 4 \times 200 \times 10^9}$$

$$= -0.0446 \text{ mm}$$

$$\text{Deflection point of B} = -0.0446 \text{ mm}$$

$$\text{Deflection point of A} = (-0.1164) + (-0.0446)$$

$$= -0.161 \text{ mm}$$

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1.12 Statically indeterminate problems.

Members for which reaction forces and internal forces can be found out from static equilibrium equations alone are called statically determinate members or structures.

Problems requiring deformation equations in addition to static equilibrium equations to solve for unknown forces are called *statically indeterminate* problems.

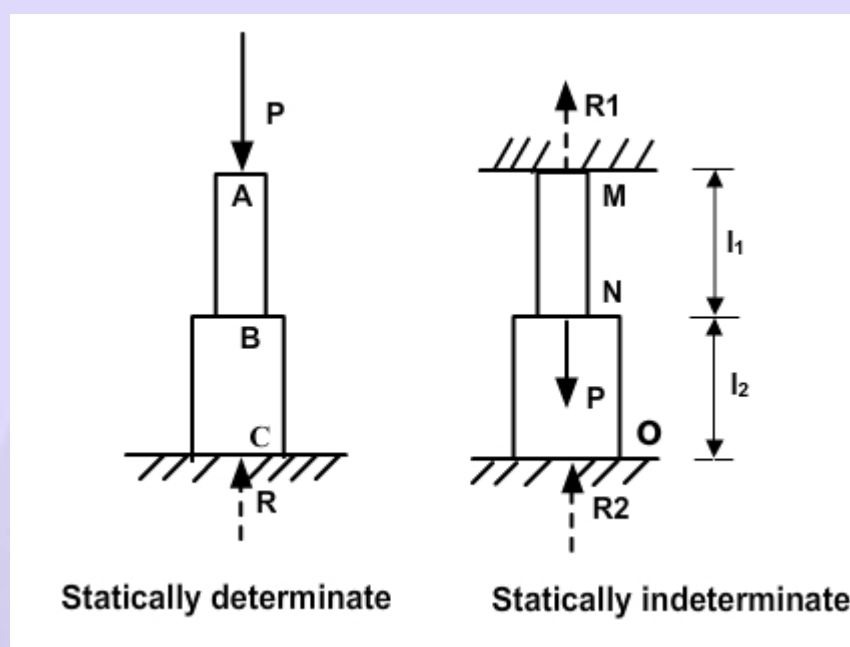


Figure 1.22

The reaction force at the support for the bar ABC in figure 1.22 can be determined considering equilibrium equation in the vertical direction.

$$\sum F_y = 0; \quad R - P = 0$$

Now, consider the right side bar MNO in figure 1.22 which is rigidly fixed at both the ends. From static equilibrium, we get only one equation with two unknown reaction forces R_1 and R_2 .

$$-P + R_1 + R_2 = 0 \quad 1.18$$

Hence, this equilibrium equation should be supplemented with a deflection equation which was discussed in the preceding section to solve for unknowns.

If the bar MNO is separated from its supports and applied the forces R_1, R_2 and P , then these forces cause the bar to undergo a deflection δ_{MO} that must be equal to zero.

$$\delta_{MO} = 0 \Rightarrow \delta_{MN} + \delta_{NO} = 0 \quad 1.19$$

δ_{MN} and δ_{NO} are the deflections of parts MN and NO respectively in the bar MNO. Individually these deflections are not zero, but their sum must make it to be zero.

Equation 1.19 is called *compatibility equation*, which insists that the change in length of the bar must be compatible with the boundary conditions.

Deflection of parts MN and NO due to load P can be obtained by assuming that the

material is within the elastic limit, $\delta_{MN} = \frac{R_1 l_1}{A_1 E}$ and $\delta_{NO} = \frac{R_2 l_2}{A_2 E}$.

Substituting these deflections in equation 1.19,

$$\frac{R_1 l_1}{A_1 E} - \frac{R_2 l_2}{A_2 E} = 0 \quad 1.20$$

Combining equations 1.18 and 1.20, one can get,

$$R_1 = \frac{P A_1 l_2}{l_1 A_2 + l_2 A_1} \quad 1.21$$

$$R_2 = \frac{P A_2 l_1}{l_1 A_2 + l_2 A_1}$$

From these reaction forces, the stresses acting on any section in the bar can be easily determined.

Example 5:

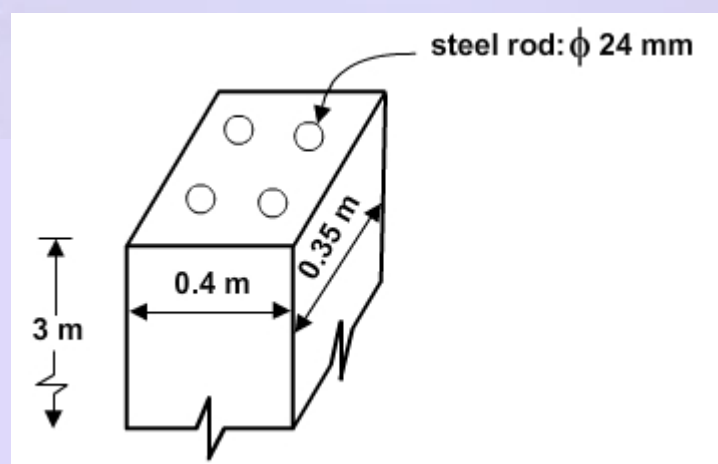


Figure 1.23

A rectangular column of sides $0.4\text{m} \times 0.35\text{m}$, made of concrete, is used to support a compressive load of 1.5MN . Four steel rods of each 24mm diameter are passing through the concrete as shown in figure 1.23. If the length of the column is 3m , determine the normal stress in the steel and the concrete. Take $E_{\text{steel}} = 200\text{ GPa}$ and $E_{\text{concrete}} = 29\text{ GPa}$.

Solution:

P_s = Load on each steel rod

P_c = Load on concrete

From equilibrium equation,

$$P_c + 4P_s = P$$

$$P_c + 4P_s = 1.5 \times 10^3 \dots\dots\dots(a)$$

Deflection in steel rod and concrete are the same.

$$\delta_{\text{concrete}} = \delta_{\text{steel}}$$

$$\frac{P_c \times 3}{(0.4) \times (0.35) \times 29 \times 10^9} = \frac{P_s \times 3}{\frac{\pi}{4} \times (0.024)^2 \times 200 \times 10^9}$$

$$P_c = 44.87P_s \dots\dots\dots(b)$$

Combining equations (a) and (b),

$$P_s = 30.7\text{kN}$$

$$P_c = 1378\text{kN}$$

$$\text{Normal stress on concrete} = \frac{P_c}{A_c} = \frac{1.378 \times 10^6}{(0.4)(0.35)} = 9.84\text{MPa}$$

$$\text{Normal stress on steel} = \frac{P_s}{A_s} = \frac{30.7 \times 10^3}{\frac{\pi}{4} (0.024)^2} = 67.86\text{MPa}$$

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1.13 Thermal effect

When a material undergoes a change in temperature, it either elongates or contracts depending upon whether heat is added to or removed from the material.

If the elongation or contraction is not restricted, then the material does not experience any stress despite the fact that it undergoes a strain.

The strain due to temperature change is called *thermal strain* and is expressed as

$$\varepsilon_T = \alpha(\Delta T) \quad 1.22$$

where α is a material property known as coefficient of thermal expansion and ΔT indicates the change in temperature.

Since strain is a dimensionless quantity and ΔT is expressed in K or $^{\circ}\text{C}$, α has a unit that is reciprocal of K or $^{\circ}\text{C}$.

The free expansion or contraction of materials, when restrained induces stress in the material and it is referred to as *thermal stress*.

Thermal stress produces the same effect in the material similar to that of mechanical stress and it can be determined as follows.

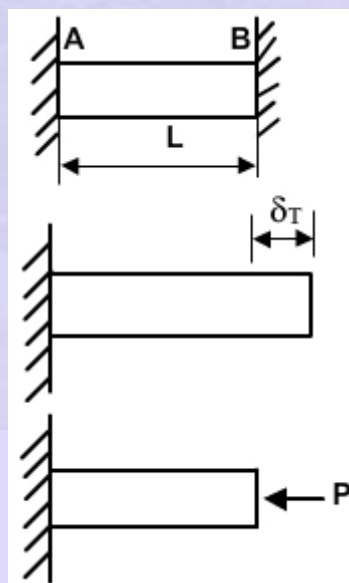


Figure 1.24

Consider a rod AB of length L which is fixed at both ends as shown in figure 1.24.

Let the temperature of the rod be raised by ΔT and as the expansion is restricted, the material develops a compressive stress.

In this problem, static equilibrium equations alone are not sufficient to solve for unknowns and hence is called statically indeterminate problem.

To determine the stress due to ΔT , assume that the support at the end B is removed and the material is allowed to expand freely.

Increase in the length of the rod δ_T due to free expansion can be found out using equation 1.22

$$\delta_T = \varepsilon_T L = \alpha(\Delta T)L \quad 1.23$$

Now, apply a compressive load P at the end B to bring it back to its initial position and the deflection due to mechanical load from equation 1.15,

$$\delta_T = \frac{PL}{AE} \quad 1.24$$

As the magnitude of δ_T and δ are equal and their signs differ,

$$\delta_T = -\delta$$

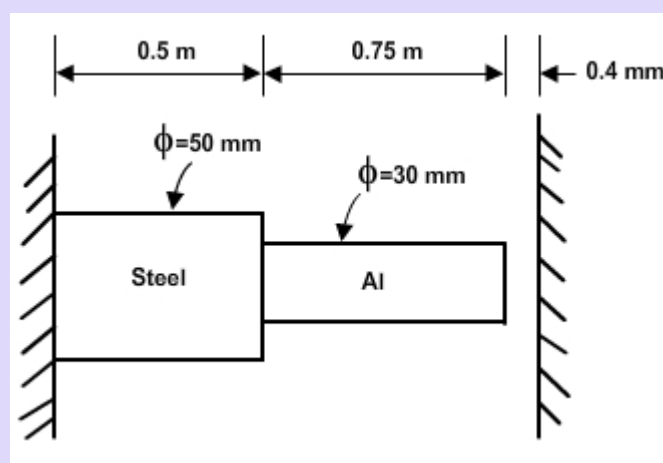
$$\alpha(\Delta T)L = -\frac{PL}{AE}$$

$$\text{Thermal stress, } \sigma_T = \frac{P}{A} = -\alpha(\Delta T)E \quad 1.25$$

Minus sign in the equation indicates a compressive stress in the material and with decrease in temperature, the stress developed is tensile stress as ΔT becomes negative.

It is to be noted that the equation 1.25 was obtained on the assumption that the material is homogeneous and the area of the cross section is uniform.

Thermoplastic analysis assumes significance for structures and components that are experiencing high temperature variations.

Example 6:**Figure 1.25**

A rod consists of two parts that are made of steel and aluminum as shown in figure 1.25. The elastic modulus and coefficient of thermal expansion for steel are 200GPa and 11.7×10^{-6} per $^{\circ}\text{C}$ respectively and for aluminum 70GPa and 21.6×10^{-6} per $^{\circ}\text{C}$ respectively. If the temperature of the rod is raised by 50°C , determine the forces and stresses acting on the rod.

Solution:

Deflection of the rod under free expansion,

$$\begin{aligned}\delta_T &= \alpha(\Delta T)L \\ &= \left(11.7 \times 10^{-6} \times 50 \times 500\right) + \left(21.6 \times 10^{-6} \times 50 \times 750\right) \\ &= 1.1025 \text{ mm}\end{aligned}$$

Restrained deflection of rod = $1.1025 - 0.4 = 0.7025 \text{ mm}$

Let the force required to make their elongation vanish be R which is the reaction force at the ends.

$$-\delta = \left(\frac{RL}{AE}\right)_{\text{steel}} + \left(\frac{RL}{AE}\right)_{\text{Al}}$$

$$\text{Area of steel rod} = \frac{\pi}{4}[0.05]^2 = 1.9635 \times 10^{-3} \text{ m}^2$$

$$\text{Area of aluminium rod} = \frac{\pi}{4}[0.03]^2 = 0.7069 \times 10^{-3} \text{ m}^2$$

$$-0.7025 = R \left[\frac{500}{1.9635 \times 10^{-3} \times 200 \times 10^9} + \frac{750}{0.7069 \times 10^{-3} \times 70 \times 10^9} \right]$$

$$\text{Compressive force on the rod, } R = -42.076 \text{ kN}$$

$$\text{Compressive stress on steel, } \sigma = \frac{P}{A} = \frac{-42.76 \times 10^3}{1.9635 \times 10^{-3}} = -21.8 \text{ MPa}$$

$$\text{Compressive stress on steel, } \sigma = \frac{P}{A} = \frac{-42.76 \times 10^3}{0.7069 \times 10^{-3}} = -60.5 \text{ MPa}$$

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1.14. Design considerations:

A good design of a structural element or machine component should ensure that the developed product will function safely and economically during its estimated life time.

The stress developed in the material should always be less than the maximum stress it can withstand which is known as ultimate strength as discussed in section 1.6.

During normal operating conditions, the stress experienced by the material is referred to as working stress or *allowable stress* or design stress.

The ratio of ultimate strength to allowable stress is defined as *factor of safety*.

$$\text{Factor of safety} = \frac{\text{Ultimate stress}}{\text{Allowable stress}} \quad 1.26$$

Factor of safety can also be expressed in terms of load as,

$$\text{Factor of safety} = \frac{\text{Ultimate load}}{\text{Allowable load}} \quad 1.27$$

Equations 1.26 and 1.27 are identical when a linear relationship exists between the load and the stress.

This is not true for many materials and equation 1.26 is widely used in design analysis.

Factor of safety take care of the uncertainties in predicting the exact loadings, variation in material properties, environmental effects and the accuracy of methods of analysis.

If the factor of safety is less, then the risk of failure is more and on the other hand, when the factor of safety is very high the structure becomes unacceptable or uncompetitive.

Hence, depending upon the applications the factor of safety varies. It is common to see that the factor of safety is taken between 2 and 3.

Stresses developed in the material when subjected to loads can be considered to be uniform at sections located far away from the point of application of loads.

This observation is called Saint Venant's principle and was discussed in section 1.3.

But, when the element has holes, grooves, notches, key ways, threads and other abrupt changes in geometry, the stress on those cross-sections will not be uniform.

These discontinuities in geometry cause high stresses concentrations in small regions of the material and are called stress raisers.

Experimentally it was found that the stress concentrations are independent of the material size and its properties, and they depend only on the geometric parameters.

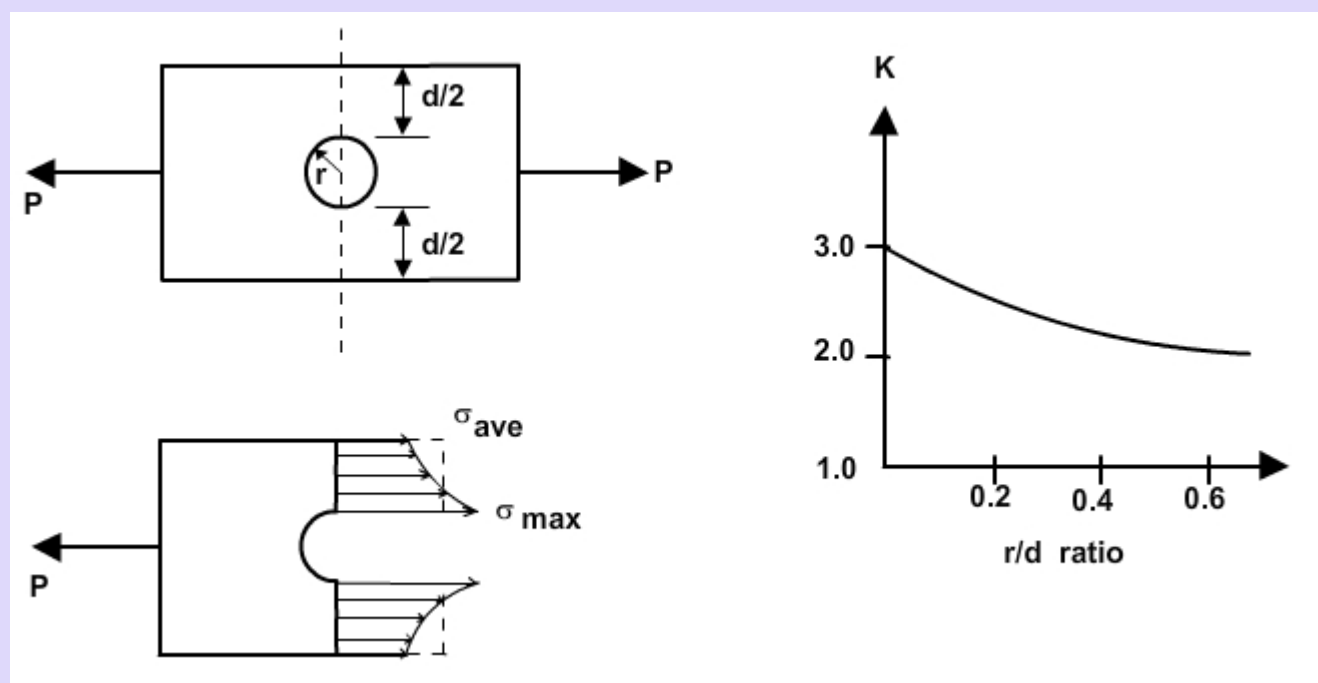


Figure 1.26

Consider a rectangular flat plate with a circular hole as shown in figure 1.26.

The stress distribution on the section passing through the centre of the hole indicates that the maximum stress occurs at the ends of the holes and it is much higher than the average stress.

Since the designer, in general, is more interested in knowing the maximum stress rather than the actual stress distribution, a simple relationship between the σ_{\max} and σ_{ave} in terms of geometric parameters will be of practical importance.

Many experiments were conducted on samples with various discontinuities and the relationship between the *stress concentration factor* and the geometrical parameters are established, where

$$\text{Stress concentration factor, } K = \frac{\sigma_{\max}}{\sigma_{\text{ave}}} \quad 1.28$$

Hence, simply by calculating the average stress, $\sigma_{ave} = \frac{P}{A}$, in the critical section of a discontinuity, σ_{max} can be easily found and by multiplying σ_{ave} with K .

The variation of K in terms of r/d for the rectangular plate with a circular hole is given in figure 1.26.

It is to be noted that the expression in equation 1.28 can be used as long as σ_{max} is within the proportional limit of the material.

Example 7:

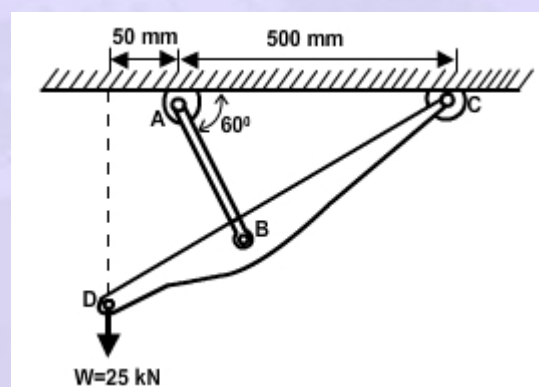
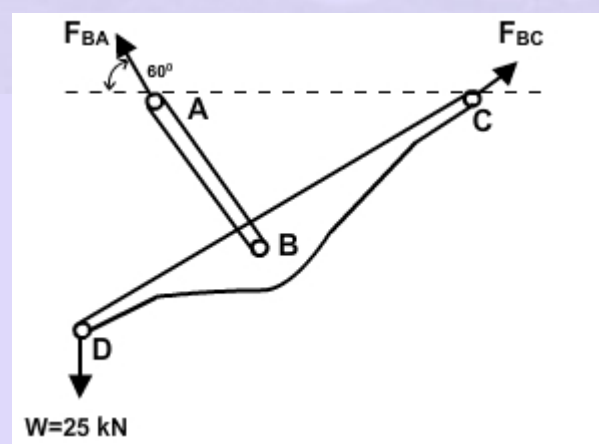


Figure 1.27

A rectangular link AB made of steel is used to support a load W through a rod CD as shown in figure 1.27. If the link AB is 30mm wide, determine its thickness for a factor of safety 2.5. The ultimate strength of steel may be assumed to be 450 MPa.

Solution:

Drawing free body diagram of the link and the rod,



Taking moment about C,

$$25 \times 550 - F_{BA} \sin 60^\circ \times 500 = 0$$

$$F_{BA} = 31.75 \text{ kN}$$

Tension along link AB, $F_{BA} = 31.75 \text{ kN}$.

$$\text{F.O.S} = \frac{\text{Ultimate stress}}{\text{allowable stress}}$$

$$\text{Allowable stress in link AB, } \sigma_a = \frac{450 \times 10^6}{2.5} = 180 \text{ MPa}$$

$$\text{Stress in link AB, } \sigma = \frac{\text{Tensile force}}{\text{Area}}$$

$$180 \times 10^6 = \frac{31.75 \times 10^3}{0.03 \times t}$$

Thickness of link AB, $t = 5.88$

$$t \approx 6 \text{ mm}$$

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1.15. Strain energy:

Strain energy is an important concept in mechanics and is used to study the response of materials and structures under static and dynamic loads.

Within the elastic limit, the work done by the external forces on a material is stored as deformation or strain that is recoverable.

On removal of load, the deformation or strain disappears and the stored energy is released. This recoverable energy stored in the material in the form of strain is called elastic strain energy.

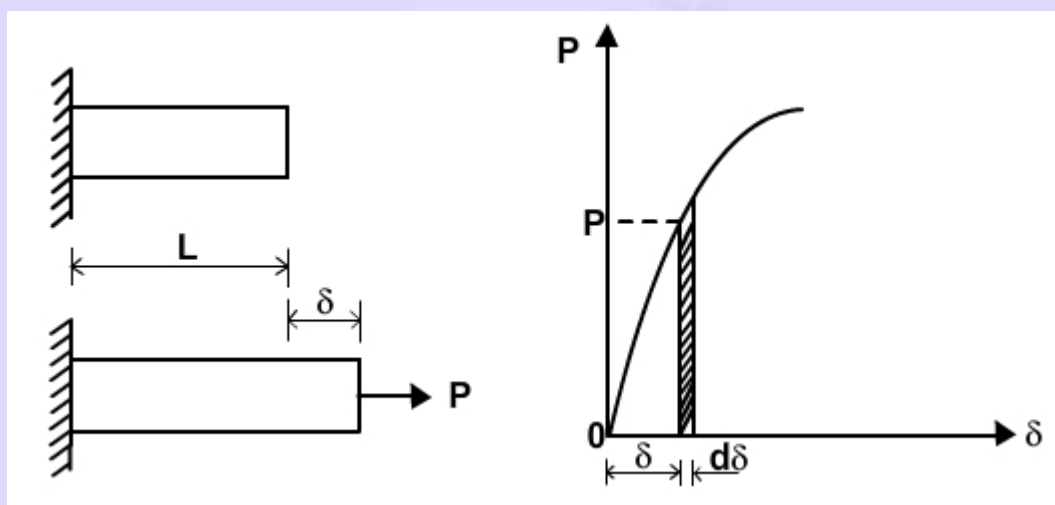


Figure 1.28

Consider a rod of uniform cross section with length L as shown in figure 1.28.

An axial tensile load P is applied on the material gradually from zero to maximum magnitude and the corresponding maximum deformation is $\bar{\delta}$.

Area under the load-displacement curve shown in figure 1.28 indicates the work done on the material by the external load that is stored as strain energy in the material.

Let dW be the work done by the load P due to increment in deflection $d\bar{\delta}$. The corresponding increase in strain energy is dU .

When the material is within the elastic limit, the work done due to $d\bar{\delta}$,

$$dW = dU = Pd\bar{\delta}$$

The total work done or total elastic strain energy of the material,

$$W = U = \int_0^{\bar{\delta}} Pd\bar{\delta} \quad 1.29$$

Equation 1.29 holds for both linear elastic and non-linear elastic materials.

If the material is linear elastic, then the load-displacement diagram will become as shown in figure 1.29.

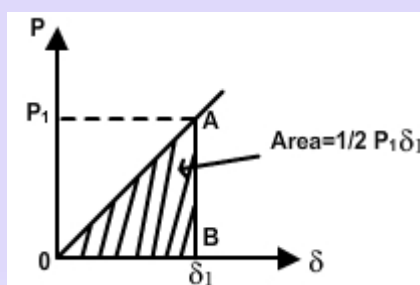


Figure 1.29

The elastic strain energy stored in the material is determined from the area of triangle OAB.

$$U = \frac{1}{2} P_1 \delta_1 \quad 1.30$$

where $\delta_1 = \frac{P_1 L}{AE}$.

Since the load-displacement curve is a straight line here, the load P_1 can be expressed in terms of stiffness and deflection as $P_1 = k\delta_1$. Then equation 1.30 turns out to be,

$$U = \frac{1}{2} k \delta_1^2 \quad 1.31$$

Work done and strain energy are expressed in N-m or joules (J).

Strain energy defined in equation 1.29 depends on the material dimensions.

In order to eliminate the material dimensions from the strain energy equation, strain energy density is often used.

Strain energy stored per unit volume of the material is referred to as *strain energy density*.

Dividing equation 1.29 by volume,

$$\text{Strain energy density, } u = \int_0^{\varepsilon} \sigma d\varepsilon \quad 1.32$$

Equation 1.32 indicates the expression of strain energy in terms of stress and strain, which are more convenient quantities to use rather than load and displacement.

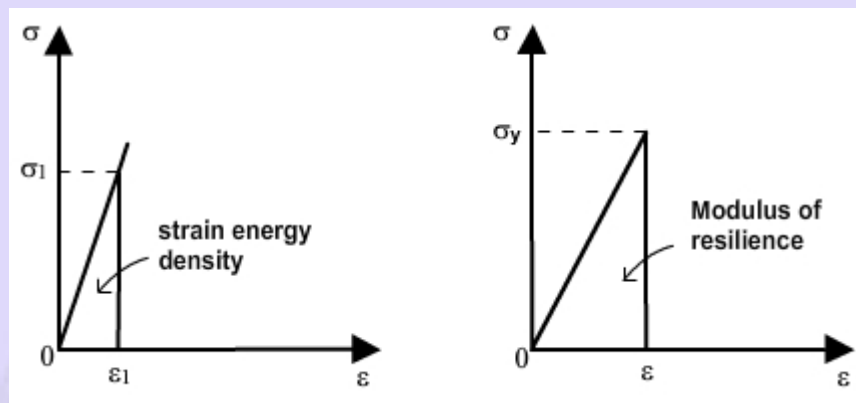


Figure 1.30

Area under the stress strain curve indicates the strain energy density of the material.

For linear elastic materials within proportional limit, equation 1.32 gets simplified as,

$$\text{Strain energy density, } u = \frac{1}{2} \sigma_1 \varepsilon_1 \quad 1.33$$

Using Hook's law, $\varepsilon_1 = \frac{\sigma_1}{E}$, strain energy density is expressed in terms of stress,

$$u = \frac{\sigma_1^2}{2E} \quad 1.34$$

When the stress in the material reaches the yield stress σ_y , the strain energy density attains its maximum value and is called the *modulus of resilience*.

$$\text{Modulus of resilience, } u_R = \frac{\sigma_Y^2}{2E} \quad 1.35$$

Modulus of resilience is a measure of energy that can be absorbed by the material due to impact loading without undergoing any plastic deformation.

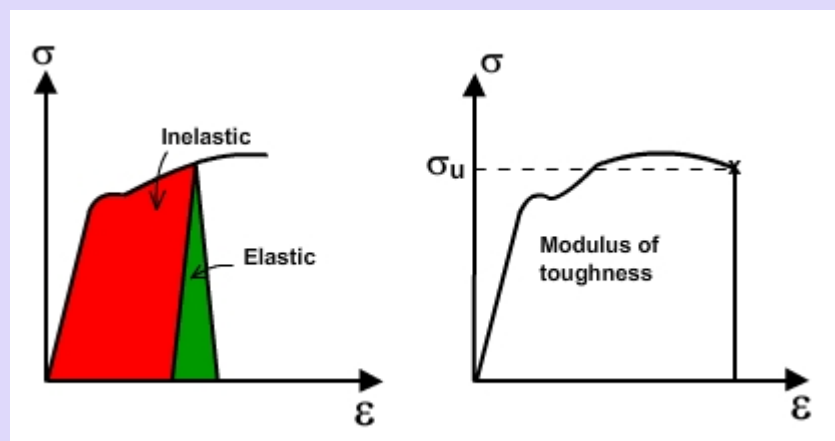


Figure 1.31

If the material exceeds the elastic limit during loading, all the work done is not stored in the material as strain energy.

This is due to the fact that part of the energy is spent on deforming the material permanently and that energy is dissipated out as heat.

The area under the entire stress strain diagram is called *modulus of toughness*, which is a measure of energy that can be absorbed by the material due to impact loading before it fractures.

Hence, materials with higher modulus of toughness are used to make components and structures that will be exposed to sudden and impact loads.

Example 8:

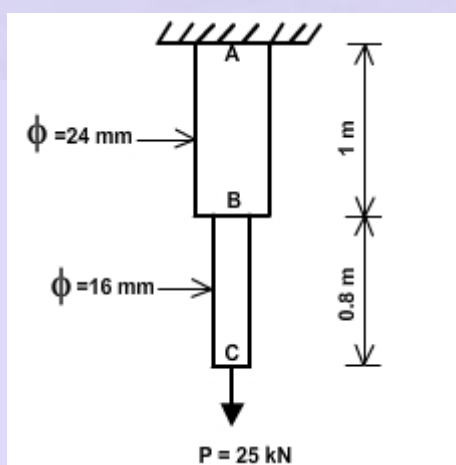


Figure 1.32

A 25 kN load is applied gradually on a steel rod ABC as shown in figure 1.32. Taking $E=200$ GPa, determine the strain energy stored in the entire rod and the strain energy density in parts AB and BC. If the yield strength of the material is $\sigma_y=320$ MPa, find the maximum energy that can be absorbed by the rod without undergoing permanent deformation.

Solution:

$$\text{Strain energy density in part AB, } u_{AB} = \frac{\sigma_{AB}^2}{2E}$$

$$u_{AB} = \frac{1}{2 \times 200 \times 10^9} \left[\frac{25 \times 10^3}{\frac{\pi}{4} (0.024)^2} \right]^2$$

$$= 7.63 \text{ kJ/m}^3$$

$$\text{Strain energy density in part BC, } u_{BC} = \frac{\sigma_{BC}^2}{2E}$$

$$u_{BC} = \frac{1}{2 \times 200 \times 10^9} \left[\frac{25 \times 10^3}{\frac{\pi}{4} (0.016)^2} \right]^2$$

$$= 38.65 \text{ kJ/m}^3$$

Strain energy in the entire rod,

$$U = u_{AB} V_{AB} + u_{BC} V_{BC}$$

$$= 7.63 \times 10^3 \times \left[\frac{\pi}{4} (0.024)^2 \times 1 \right] + 38.65 \times 10^3 \times \left[\frac{\pi}{4} (0.016)^2 \times 0.8 \right]$$

$$U = 9.67 \text{ J}$$

The load that will produce yield stress in the material,

$$P = \sigma_y A_{BC} = 320 \times 10^6 \times \frac{\pi}{4} (0.016)^2$$

$$P = 64.3 \text{ kN}$$

Maximum energy that can be stored in the rod,

$$\begin{aligned}U &= \frac{1}{2E} \left[\left(\frac{P}{A_{AB}} \right)^2 \times V_{AB} + \left(\frac{P}{A_{BC}} \right)^2 \times V_{BC} \right] \\&= \frac{P^2}{2E} \left[\frac{L_{AB}}{A_{AB}} + \frac{L_{BC}}{A_{BC}} \right] \\&= \frac{(64.3 \times 10^3)^2}{2 \times 200 \times 10^9} \left[\frac{1}{\frac{\pi}{4} (0.024)^2} + \frac{0.8}{\frac{\pi}{4} (0.016)^2} \right] \\&= 63.97 \text{ J}\end{aligned}$$

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1.16 Impact loading

A static loading is applied very slowly so that the external load and the internal force are always in equilibrium. Hence, the vibrational and dynamic effects are negligible in static loading.

Dynamic loading may take many forms like fluctuating loads where the loads are varying with time and impact loads where the loads are applied suddenly and may be removed immediately or later.

Collision of two bodies and objects freely falling onto a structure are some of the examples of impact loading.

Consider a collar of mass M at a height h from the flange that is rigidly fixed at the end of a bar as shown in figure 1.33.

As the collar freely falls onto the flange, the bar begins to elongate causing axial stresses and strain within the bar.

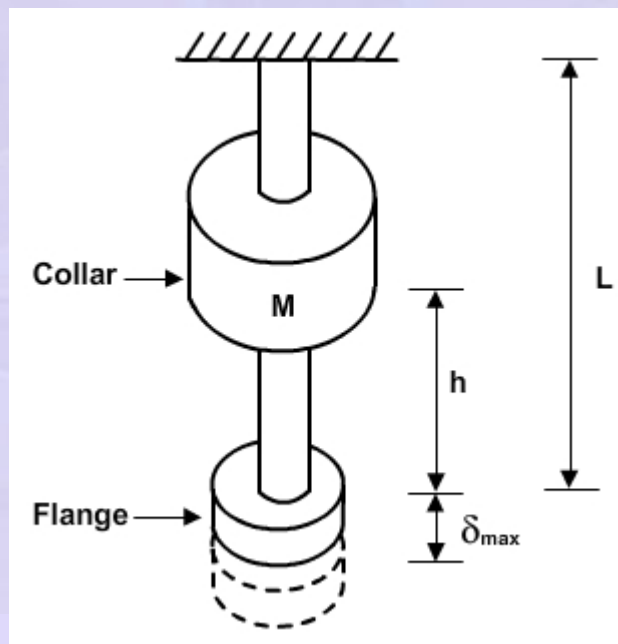


Figure 1.33

After the flange reaching its maximum position during downward motion, it moves up due to shortening of the bar.

The bar vibrates in the axial direction with the collar and the flange till the vibration dies out completely due to damping effects.

To simplify the complex impact loading analysis, the following assumptions are made.

The kinetic energy of the collar at the time of striking is completely transformed into strain energy and stored in the bar.

But in practice, not all the kinetic energy is stored in the material as some of the energy is dissipated out as heat and waves.

Hence, this assumption is conservative in the sense that the stress and deflection predicted by this way are higher than the actual values.

The second assumption is that after striking the flange, the collar and the flange move downward together without any bouncing.

This assumption is reasonable provided the weight of the collar is much larger than that of the bar.

The third assumption is that the stresses in the bar remain within linear elastic range and the stress distribution is uniform within the bar.

But, in reality, the stress distribution is not uniform since the stress waves generated due to impact loading travel through the bar.

Using the principle of conservation of energy, the kinetic energy of the collar is equated to the strain energy of the bar.

Assuming the height of fall h is much larger than the deformation of rod, and using equation 1.34,

$$\frac{1}{2}Mv^2 = \frac{\sigma_{\max}^2 V}{2E} \quad 1.36$$

where v is the velocity of the collar at strike ($v = \sqrt{2gh}$) and V is the volume of the material.

The maximum stress in the bar due to the impact load of mass M ,

$$\sigma_{\max} = \sqrt{\frac{Mv^2 E}{V}} \quad 1.37$$

From above equation, it becomes clear that by increasing the volume of material, the effect of impact loading can be minimized.

Expressing strain energy in terms of deflection in equation 1.36,

$$\frac{1}{2}Mv^2 = \frac{\delta_{\max}^2 EA}{2L}$$

$$\delta_{\max} = \sqrt{\frac{Mv^2 L}{EA}} \quad 1.38$$

If the load of the collar is applied gradually on the bar i.e., under static loading, the static deflection δ_{st} will be,

$$\delta_{\text{st}} = \frac{MgL}{EA}$$

Substituting this in equation 1.38, relationship between the static deflection δ_{st} and the impact deflection δ_{\max} is obtained.

$$\delta_{\max} = \sqrt{2h\delta_{\text{st}}} \quad 1.39$$

To represent the magnification of deflection due to impact load compared to that of static deflection for the same load, *impact factor* is used.

$$\text{Impact factor} = \frac{\delta_{\max}}{\delta_{\text{st}}} \quad 1.40$$

Alternately, the impact factor can be obtained from the ratio $\frac{\sigma_{\max}}{\sigma_{\text{st}}}$.

The relationship between the stress σ_{st} developed in the bar due to static loading and the impact loading stress σ_{\max} is determined as follows.

$$\sigma_{\text{st}} = E\varepsilon_{\text{st}} = \frac{E\delta_{\text{st}}}{L}$$

$$\sigma_{\max} = E\varepsilon_{\max} = \frac{E\delta_{\max}}{L}$$

$$\sigma_{\max} = \frac{E}{L} \sqrt{2h\delta_{\text{st}}}$$

$$\sigma_{\max} = \sqrt{\frac{2hE\sigma_{\text{st}}}{L}} \quad 1.41$$

Now, the effect of suddenly applied loads on materials or structures that forms a special case of impact loading is discussed.

In figure 1.33, if the collar is brought into contact on the top of the flange and released immediately, it is referred to as suddenly applied load.

The maximum stress produced in the bar due to suddenly applied load can be determined by replacing h by δ_{\max} in equation 1.41

$$\sigma_{\max} = 2\sigma_{st} \quad 1.42$$

Hence, the stress developed in a material due to suddenly applied load is twice as large as that of gradually applied load.

Example 9:

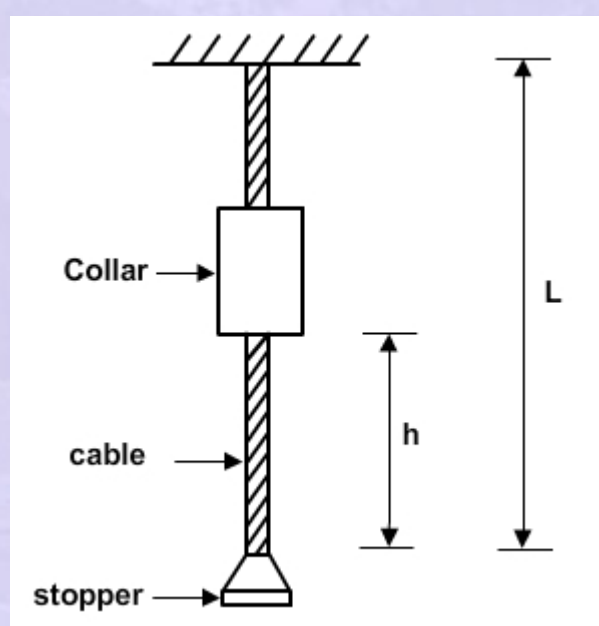


Figure 1.34

A 50 kg collar is sliding on a cable as shown in figure 1.34 from a height $h = 1\text{m}$. Its free fall is restrained by a stopper at the end of the cable. The effective cross-sectional area and the elastic modulus of the cable are taken to be 60 mm^2 and 150GPa respectively. If the maximum allowable stress in the cable due to impact load is 450MPa , calculate the minimum permissible length for the cable and the corresponding maximum deflection. Also find the impact factor.

Solution:

Maximum stress due to impact load, $\sigma_{\max} = \sqrt{\frac{Mv^2 E}{V}}$

$$\text{Velocity, } v = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 1} = 4.43 \text{ m/s}$$

$$450 \times 10^6 = \sqrt{\frac{50 \times (4.43)^2 \times 150 \times 10^9}{(60 \times 10^{-6}) \times L}}$$

Minimum permissible length for the cable, $L = 12.1 \text{ m}$

$$\begin{aligned} \text{Static deflection, } \delta_{st} &= \frac{MgL}{EA} \\ &= \frac{50 \times 9.81 \times 12.1}{150 \times 10^9 \times 60 \times 10^{-6}} \\ &= 0.656 \text{ mm} \end{aligned}$$

$$\begin{aligned} \text{Maximum deflection, } \delta_{max} &= \sqrt{2h\delta_{st}} \\ &= \sqrt{2 \times 1000 \times 0.656} \\ &= 36 \text{ mm} \end{aligned}$$

$$\begin{aligned} \text{Impact factor, } &= \frac{\delta_{max}}{\delta_{st}} = \frac{36}{0.656} \\ &= 55 \end{aligned}$$



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