Boundary Layer Theory (Lectures 37 to 40)

Q1. Choose the correct answer

(i) The boundary layer thickness for flow over a flat plate

(a) increases with an increase in the free stream velocity
(b) decreases with an increase in the free stream velocity
(c) increases with an increase in the kinematic viscosity
(d) decreases with an increase in the kinematic viscosity

[*Ans.* (b) and (c)]

(ii) If x is the distance measured from the leading edge of a flat plate, the wall shear stress for laminar boundary layer varies as

(a) x
(b) $x^{1/2}$
(c) $x^{-1/2}$
(d) $x^{-4/5}$

[*Ans.*(b)]

(iii) The growth of the turbulent boundary layer thickness as compared to the laminar boundary layer takes place

(a) at a slower rate
(b) at a faster rate
(c) at the same rate
(d) unpredictable

[*Ans.*(b)]

(iv) Separation in flow past a solid object is caused by

(a) a favourable (negative) pressure gradient
(b) an adverse (positive) pressure gradient
(c) the boundary layer thickness reducing to zero
(d) a reduction of pressure to vapour pressure

[*Ans.*(b)]

Q2.

The velocity profile over a flat plate of length L is expressed in terms of a similarity variable η as $\frac{u}{U_\infty} = \frac{dF(\eta)}{d\eta}$, where $\eta = y\sqrt{\frac{U_\infty}{\upsilon x}}$. Numerical solution for the variable reveals that $\left.\frac{d^2F}{d\eta^2}\right|_{\eta=0} = 0.3$. Determine the drag coefficient as a function of Reynolds number, Re_L.

Solution

The wall shear stress is found to be

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

$$= \mu \left.\frac{\partial u}{\partial \eta}\right|_{\eta=0} \frac{\partial \eta}{\partial y}$$

$$= \mu U_\infty \left.\frac{d^2F}{d\eta^2}\right|_{\eta=0} \sqrt{\frac{U_\infty}{\upsilon x}}$$

$$= 0.3\mu U_\infty \sqrt{\frac{U_\infty}{\upsilon x}}$$

The drag force acting on one side of the plate is

$F_D = \int_0^L \tau_w w dx$ (w is the width of the plate in a direction perpendicular to the plate)

$$= \int_0^L 0.3\mu U_\infty \sqrt{\frac{U_\infty}{\upsilon x}} w dx$$

$$= 0.6 U_\infty w \mu \sqrt{\mathrm{Re}_L}$$

The drag coefficient is then

$$C_{fL} = \frac{F_D}{\frac{1}{2}\rho U_\infty^2 w L} = \frac{0.6 U_\infty w \mu \sqrt{\mathrm{Re}_L}}{\frac{1}{2}\rho U_\infty^2 w L} = \frac{1.2}{\sqrt{\mathrm{Re}_L}}$$

Q3.

The velocity profile within boundary layer for steady, two-dimensional, constant density, laminar flow over a flat plate is a polynomial of order 4 and is given as:

$$\frac{u}{U_\infty} = C_0 + C_1 \frac{y}{\delta} + C_2 \left(\frac{y}{\delta}\right)^2 + C_3 \left(\frac{y}{\delta}\right)^3 + C_4 \left(\frac{y}{\delta}\right)^4$$

Using suitable boundary conditions, evaluate the constants C_0, C_1, C_2 , C_3 and C_4.

Solution

The boundary conditions are

(i) at $y = 0$, $u = 0$

(ii) at $y = \delta$, $u = U_\infty$

(iii) at $y = \delta$, $\frac{\partial u}{\partial y} = 0$

(iv)at $y = 0$, $\frac{\partial^2 u}{\partial y^2} = 0$

(v) at $y = \delta$, $\frac{\partial^2 u}{\partial y^2} = 0$

Applying $u = 0$ at $y = 0$, one can get

$$0 = C_0$$

Applying $u = U_\infty$ at $y = \delta$ gives

$$1 = C_1 + C_2 + C_3 + C_4 \qquad (1)$$

Differentiating the velocity profile (note that both u and δ are functions of x only) with respect to y, we obtain

$$\frac{1}{U_\infty}\frac{\partial u}{\partial y}=\frac{C_1}{\delta}+\frac{2C_2}{\delta}\frac{y}{\delta}+\frac{3C_3}{\delta}\left(\frac{y}{\delta}\right)^2+\frac{4C_4}{\delta}\left(\frac{y}{\delta}\right)^3$$

Applying (iii) boundary condition i.e., $\frac{\partial u}{\partial y}=0$ at $y=\delta$, we get

$$0=C_1+2C_2+3C_3+4C_4 \qquad (2)$$

A second differentiation of the velocity profile gives

$$\frac{1}{U_\infty}\frac{\partial^2 u}{\partial y^2}=\frac{2C_2}{\delta^2}+\frac{6C_3}{\delta^2}\frac{y}{\delta}+\frac{12C_4}{\delta^2}\left(\frac{y}{\delta}\right)^2$$

Applying (iv) boundary condition i.e., $\frac{\partial^2 u}{\partial y^2}=0$ at $y=0$, we obtain

$$0=C_2 \qquad (3)$$

Applying (v) boundary condition i.e., $\frac{\partial^2 u}{\partial y^2}=0$ at $y=\delta$, we obtain

$$0=C_3+2C_4 \qquad (4)$$

Solving Eqs (1), (2), (3) and (4) simultaneously, we obtain

$$C_1=2,\ C_3=-2 \text{ and } C_4=1$$

Then the velocity profile becomes

$$\frac{u}{U_\infty}=2\frac{y}{\delta}-2\left(\frac{y}{\delta}\right)^3+\left(\frac{y}{\delta}\right)^4$$

Q4.

The most general sinusoidal velocity profile for laminar boundary-layer flow on a flat plate is $u=A\sin(By)+C$. State three boundary conditions applicable to the laminar boundary-layer velocity profile. Evaluate constants *A*, *B* and *C*.

For the above velocity profile find expressions for:

(a) the rate of growth of δ as a function of x.

(b) the local skin friction coefficient in terms of distance and flow properties.

(c) the total drag force on a plate of length L and width *w*.

Solution

The velocity profile is given as

$$u=A\sin(By)+C$$

The boundary conditions are

(i) at $y=0$, $u=0$ (no-slip condition at the plate)

(ii) at $y=\delta$, $u=U_\infty$ (free stream velocity at the edge of the boundary layer)

(iii) at $y=\delta$, $\frac{\partial u}{\partial y}=0$ (zero shear stress at the edge of the boundary layer)

Applying $u=0$ at $y=0$, one can get

$$0=C$$

Applying $u=U_\infty$ at $y=\delta$ gives

$$U_\infty = A\sin(B\delta) \qquad (1)$$

Applying (iii) boundary condition i.e., $\frac{\partial u}{\partial y} = 0$ at $y = \delta$, we get

$$0 = AB\cos(B\delta)$$

or $$B\delta = \frac{\pi}{2}$$

or $$B = \frac{\pi}{2\delta}$$

Substituting $B = \frac{\pi}{2\delta}$ in Eq. (1), we obtain

$$U_\infty = A\sin(\frac{\pi}{2\delta}\delta) = A\sin(\frac{\pi}{2}) = A$$

or $$A = U_\infty$$

The velocity profile is then

$$u = U_\infty \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

Substituting $\frac{u}{U_\infty} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$ into *von Karman integral equation* ($\frac{\tau_w}{\rho U_\infty^2} = \frac{d}{dx}\int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)dy$), we obtain

$$\frac{\tau_w}{\rho U_\infty^2} = \frac{d}{dx}\int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)dy$$

$$= \frac{d}{dx}\int_0^\delta \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\left[1 - \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right]dy$$

$$= \frac{d}{dx}\int_0^\delta \left[\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - \sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right]dy$$

$$= \frac{d}{dx}\int_0^\delta \left[\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - \frac{1}{2}\left\{1 - \cos\left(\pi\frac{y}{\delta}\right)\right\}\right]dy \qquad \left[\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$$

$$= \frac{d}{dx}\int_0^\delta \left[\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - \frac{1}{2} + \frac{1}{2}\cos\left(\pi\frac{y}{\delta}\right)\right]dy$$

$$= \frac{d}{dx}\left[\frac{-\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)}{\frac{\pi}{2\delta}} - \frac{1}{2}y + \frac{1}{2}\frac{\sin\left(\pi\frac{y}{\delta}\right)}{\frac{\pi}{\delta}}\right]_0^\delta$$

$$= \frac{d}{dx}\left[\frac{-\cos\left(\frac{\pi}{2}\frac{\delta}{\delta}\right)}{\frac{\pi}{2\delta}} + \frac{\cos\left(\frac{\pi}{2}\times\frac{0}{\delta}\right)}{\frac{\pi}{2\delta}} - \frac{1}{2}\delta - \frac{1}{2}\times 0 + \frac{\sin\left(\pi\frac{\delta}{\delta}\right)}{\frac{2\pi}{\delta}} - \frac{\sin\left(\pi\times\frac{0}{\delta}\right)}{\frac{2\pi}{\delta}} \right]$$

$$= \frac{d}{dx}\left[0 + \frac{1}{\frac{\pi}{2\delta}} - \frac{1}{2}\delta + 0 - 0 \right]$$

$$= \frac{d}{dx}\left[\frac{2\delta}{\pi} - \frac{\delta}{2} \right]$$

$$= \frac{d\delta}{dx}\left[\frac{2}{\pi} - \frac{1}{2} \right] = 0.137\frac{d\delta}{dx}$$

or $$\tau_w = 0.137\rho U_\infty^2 \frac{d\delta}{dx} \qquad (2)$$

The velocity at a location *y* measured from the plate is $u = U_\infty \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$

Differentiating with respect to *y*, we have

$$\frac{\partial u}{\partial y} = U_\infty \frac{\pi}{2\delta}\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

At $y = 0$, $$\left.\frac{\partial u}{\partial y}\right|_{y=0} = U_\infty \frac{\pi}{2\delta}\cos\left(\frac{\pi}{2}\times\frac{0}{\delta}\right) = \frac{\pi U_\infty}{2\delta}$$

Therefore, the wall shear stress becomes

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu\frac{\pi U_\infty}{2\delta} = \frac{\mu\pi U_\infty}{2\delta} \qquad (3)$$

Equating the values of τ_w from Eqs (2) and (3), we obtain

$$\tau_w = 0.137\rho U_\infty^2 \frac{d\delta}{dx} = \frac{\pi\mu U_\infty}{2\delta}$$

or, $$\delta\frac{d\delta}{dx} = \frac{\pi\mu}{0.274\rho U_\infty}$$

or, $$\delta d\delta = 11.466\frac{\mu}{\rho U_\infty}dx$$

Integrating the above equation, we get

$$\frac{\delta^2}{2} = 11.466\frac{\mu}{\rho U_\infty}x + C$$

At $x = 0$, $\delta = 0$, so $C = 0$.
Then

$$\frac{\delta^2}{2} = 11.466\frac{\mu}{\rho U_\infty}x$$

or, $$\delta^2 = \frac{22.932\mu}{\rho U_\infty} x$$

or, $$\delta = \sqrt{\frac{22.932\mu}{\rho U_\infty} x} = 4.79\sqrt{\frac{\mu}{\rho U_\infty} x}$$

or, $$\frac{\delta}{x} = 4.79\sqrt{\frac{\mu}{\rho U_\infty x}} = \frac{4.79}{\sqrt{\mathrm{Re}_x}}$$

The local *skin friction coefficient* C_{fx} is given by

$$C_{fx} = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2}$$

$$= \frac{\frac{\mu\pi U_\infty}{2\delta}}{\frac{1}{2}\rho U_\infty^2} = \frac{\mu\pi}{\rho\delta U_\infty} = \frac{\mu\pi}{\rho U_\infty \times 4.79\sqrt{\frac{\mu}{\rho U_\infty} x}}$$

$$= \frac{0.656}{\sqrt{\mathrm{Re}_x}}$$

The total shear force F_D on one side of the plate is given by

$$F_D = \int_0^L \tau_w w dx$$

$$= \int_0^L \frac{\pi\mu U_\infty}{2\delta} w dx$$

$$= \int_0^L \frac{\pi\mu U_\infty}{2 \times 4.79\sqrt{\frac{\mu}{\rho U_\infty} x}} w dx$$

$$= 0.327\rho U_\infty^2 w\sqrt{\frac{\mu}{\rho U_\infty}} \int_0^L \sqrt{\frac{1}{x}} dx$$

$$= 0.327\rho U_\infty^2 w\sqrt{\frac{\mu}{\rho U_\infty}} \left[2x^{\frac{1}{2}}\right]_0^L$$

$$= 0.655\rho U_\infty^2 w\sqrt{\frac{\mu}{\rho U_\infty}} L^{\frac{1}{2}}$$