Dynamics of Viscous Flows (Lectures 28 to 32)

Q1. **Choose the correct answer**

(i) The average velocity of a one-dimensional incompressible fully developed viscous flow between two fixed parallel plates is 2 m/s. The maximum velocity of the flow is

(a) 2 m/s

(b) 3 m/s

(c) 4 m/s

(d) 5 m/s

[*Ans.(*b)]

(ii) Which of the following statements is correct for a fully developed pipe flow?

(a) pressure gradient is greater than the wall shear stress

(b) the velocity profile is changing continuously

(c) inertia force balances the wall shear stress

(d) pressure gradient balances the wall shear stress only and has a constant value.

[*Ans.(*d)]

(iii) The velocity profile of a fully developed laminar flow in a straight circular pipe is given by $v_z = -\frac{R^2}{4\mu}\frac{dp}{dz}\left(1-\frac{r^2}{R^2}\right)$, where $\frac{dp}{dz}$ is the constant , R is the radius of the pipe and r is the radial distance from the centre of the pipe. The maximum velocity of fluid in the pipe is

(a) $-\frac{R^2}{\mu}\frac{dp}{dz}$

(b) $-\frac{R^2}{2\mu}\frac{dp}{dz}$

(c) $-\frac{R^2}{4\mu}\frac{dp}{dz}$

(d) $-\frac{R^2}{8\mu}\frac{dp}{dz}$

[*Ans.(*c)]

(iv) For fully developed, laminar flow through circular pipes Darcy friction factor (*f*) is given by

(a) $f = \frac{16}{\mathrm{Re}}$

(b) $f = \frac{64}{\mathrm{Re}}$

(c) $f = \frac{4}{\mathrm{Re}}$

(d) $f = \frac{32}{\mathrm{Re}}$

[*Ans.(*b)]

Q2.

Water at $20°C$ flows between two large parallel plates which are 2 mm apart. If the maximum velocity is 1.5 m/s, determine the average velocity, pressure drop per unit length and the shear stress at walls of the plate. Viscosity of water at $20°C$ is 0.001 Pa-s.

Solution

The average velocity is given by

$$\bar{u} = \frac{2}{3}u_{\max} = \frac{2}{3} \times 1.5 = 1 \text{ m/s}$$

The pressure drop in a two dimensional straight channel is given by

$$\frac{dp}{dx} = \frac{-2\mu u_{\max}}{H^2}, \text{ where } H = \text{half the channel height}$$

Putting the respective values, we get

$$\frac{dp}{dx} = \frac{-2 \times 0.001 \times 1.5}{0.001^2}$$

$$= -3000 \text{ N/m}^3$$

$$= -3 \text{ kN/m}^2 \text{ per m}$$

The wall shearing stress in a channel flow is given by

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{\text{wall}} = -H\frac{dp}{dx} = -0.001 \times (-3000) = 3 \text{ N/m}^2$$

Q3.

Two viscous, incompressible, immiscible fluids of same density ($=\rho$) but different viscosities (viscosity of the lower fluid layer $=\mu_1$ and that of the upper fluid layer $=\mu_2 < \mu_1$) flow in separate layers between parallel boundaries located at $y = \pm H$ as shown in the figure below. Thickness of each fluid layer is identical and their interface is flat. The flow is driven by a constant favourable pressure gradient of $\frac{dp}{dx}$. Derive expressions for the velocity profiles in the fluid layers. Assume the flow to be steady and the plates to be of infinitely large width.

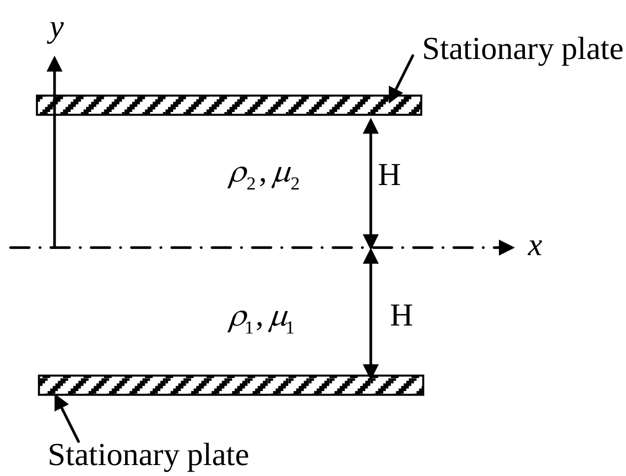


Fig.

Solution

Large lateral width of the plate renders the basic flow consideration to be two dimensional, for which the continuity equation under incompressible flow conditions reads:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

For fully developed flow, $\frac{\partial u}{\partial x}=0$

Hence, $$\frac{\partial v}{\partial y}=0$$

$$\Rightarrow v\neq v(y)$$

Since $v=0$ at $y=\pm H$ as a result of the no-penetration at the walls, v is identically equal to zero for all y, i.e.,

$$\therefore v=0 \text{ at } -H\leq y\leq H$$

Now, considering x-momentum equation, we have, for steady flow,

$$\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]=-\frac{dp}{dx}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Under assumptions mentioned as above, the above equation becomes

$$\underbrace{\frac{d^2u}{dy^2}}_{\text{Function of y only}} \quad = \quad \underbrace{\frac{1}{\mu}\frac{dp}{dx}}_{\text{Function of x only}} \quad = \text{ constant}$$

For $-H\leq y\leq 0$

$$\frac{d^2u_1}{dy^2}=\frac{1}{\mu_1}\frac{dp}{dx}$$

Integrating the above equation, we have

$$\frac{du_1}{dy}=\frac{1}{\mu_1}\frac{dp}{dx}y+C_1$$

$$u_1=\frac{1}{\mu_1}\frac{dp}{dx}\frac{y^2}{2}+C_1y+C_2 \qquad (1)$$

where C_1 and C_2 are constants of integration .

For $0\leq y\leq H$

$$\frac{d^2u_2}{dy^2}=\frac{1}{\mu_2}\frac{dp}{dx}$$

Integrating the above equation, we have

$$\frac{du_2}{dy}=\frac{1}{\mu_2}\frac{dp}{dx}y+C_3$$

$$u_2=\frac{1}{\mu_2}\frac{dp}{dx}\frac{y^2}{2}+C_3y+C_4 \qquad (2)$$

where C_3 and C_4 are constants of integration.

Eqs (1) and (2) are subjected to the following boundary conditions

At $y = -H$, $u_1 = 0$,

At $y = H$, $u_2 = 0$,

At $y = 0$, $u_1 = u_2$ (continuity of flow velocity)

At $y = 0$, $\mu_1 \frac{du_1}{dy} = \mu_2 \frac{du_2}{dy}$ (continuity of shear stress)

From the boundary conditions at $y = -H$, $u_1 = 0$, and at $y = H$, $u_2 = 0$, we get

$$C_2 = C_1 H - \frac{1}{\mu_1} \frac{dp}{dx} \frac{H^2}{2}$$

$$C_4 = -C_3 H - \frac{1}{\mu_2} \frac{dp}{dx} \frac{H^2}{2}$$

From the boundary condition at $y = 0$, $\mu_1 \frac{du_1}{dy} = \mu_2 \frac{du_2}{dy}$, we have

$$\mu_1 C_1 = \mu_2 C_3$$

$$\Rightarrow C_1 = \frac{\mu_2}{\mu_1} C_3$$

From the boundary condition at $y = 0$, $u_1 = u_2$, one can write

$$C_2 = C_4$$

or, $$C_1 H - \frac{1}{\mu_1} \frac{dp}{dx} \frac{H^2}{2} = -C_3 H - \frac{1}{\mu_2} \frac{dp}{dx} \frac{H^2}{2}$$

or, $$\left(C_1 + C_3\right) H = \frac{dp}{dx} \frac{H^2}{2} \left[\frac{1}{\mu_1} - \frac{1}{\mu_2} \right]$$

or, $$\left(\frac{\mu_2}{\mu_1} + 1 \right) C_3 = \frac{dp}{dx} \frac{H}{2} \left[\frac{1}{\mu_1} - \frac{1}{\mu_2} \right]$$

or, $$C_3 = \frac{dp}{dx} \frac{H}{2} \frac{\left(\mu_2 - \mu_1\right)}{\left(\mu_2 + \mu_1\right) \mu_2}$$

$$C_1 = \frac{dp}{dx} \frac{H}{2} \frac{\left(\mu_2 - \mu_1\right)}{\left(\mu_2 + \mu_1\right) \mu_1}$$

Substituting the values of C_1 , C_2 , C_3 and C_4 in the Eqs (1) and (2), we have

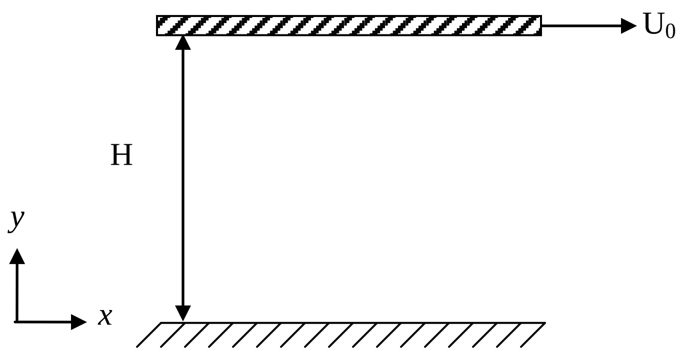
$$u_1 = \frac{1}{\mu_1} \frac{dp}{dx} \frac{1}{2} \left[y^2 - H^2 + \frac{\left(\mu_2 - \mu_1\right)}{\left(\mu_2 + \mu_1\right)} H \left(y + H\right) \right]$$

$$u_2 = \frac{1}{\mu_2} \frac{dp}{dx} \frac{1}{2} \left[y^2 - H^2 + \frac{\left(\mu_2 - \mu_1\right)}{\left(\mu_2 + \mu_1\right)} H \left(y - H\right) \right]$$

Q4.

Consider steady, incompressible, fully developed flow between parallel plates H distance apart. The upper plate moves to the right at a uniform speed of U_0, while the lower plate is stationary. Determine the pressure gradient required to produce zero net flow at a section.

Solution



The governing equation for the motion of flow can be written as

$$\mu \frac{d^2u}{dy^2} = \frac{dp}{dx}$$

Integrating the above equation twice with respect to y, we obtain

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx}y + C_1$$

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + C_1 y + C_2$$

where C_1 and C_2 are constants of integration.

To evaluate the constants, we apply the boundary conditions. The boundary conditions are:

at $y = 0,\quad u = 0$

at $y = H,\quad u = U_0$

Applying the boundary conditions, we get

$$C_1 = \frac{U_0}{H} - \frac{1}{2\mu}\frac{dp}{dx}H \text{ and } C_2 = 0$$

Therefore, we obtain

$$u = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - yH\right) + \frac{U_0}{H}y$$

The volume flow rate is given by

$$Q = \int_0^H uw\,dy \qquad \left[w = \text{width of the plate}\right]$$

$$= w\int_0^H \left[\frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - Hy\right) + U_0\frac{y}{H}\right]dy$$

$$= w\left(-\frac{1}{12\mu}\frac{dp}{dx}H^3 + \frac{U_0 H}{2}\right)$$

For zero flow rate ($Q = 0$), we have

$$w\left(-\frac{1}{12\mu}\frac{dp}{dx} + \frac{U_0 H}{2}\right) = 0$$

or

$$\frac{dp}{dx} = \frac{6\mu U_0}{H^2}$$

Q5.

An oil of density 800 kg/m^3 and viscosity $0.12\ \mathrm{Ns/m^2}$ is flowing through a pipe of diameter 20 cm at the rate of 15 litre/s. Find the head lost due to friction and power required to maintain the flow for a length of 500m.

Solution

Average velocity of flow is

$$\bar{u} = \frac{Q}{A} = \frac{Q}{\frac{\pi}{4}D^2} = \frac{0.015}{\frac{\pi}{4}\times 0.2^2} = 0.478\ \mathrm{m/s}$$

Reynolds number is

$$\mathrm{Re} = \frac{\rho \bar{u} D}{\mu}$$

$$= \frac{800\times 0.478\times 0.2}{0.12} = 637.33$$

The flow is laminar since $\mathrm{Re} < 2000$.

Head lost due to friction is given by

$$h_f = \frac{132\mu Q L}{\pi \rho g D^4}$$

$$= \frac{132\times 0.12\times 0.015\times 500}{\pi\times 800\times 9.81\times 0.2^4} = 3.01\ \mathrm{m}$$

The power required to maintain the flow is

$$P = \rho g Q h_f$$

$$= 800\times 9.81\times 0.015\times 3.01$$

$$= 354.34\ \mathrm{W}$$