Dynamics of Inviscid Flows (Lectures 16 to 20)

Q1. **Choose the correct answer**

(i) Euler's equation is written as

(a) $\rho\left[\dfrac{\partial \vec{V}}{\partial t}+\left(\vec{V}.\nabla\right)\vec{V}\right]=\nabla p+\rho\vec{b}$

(b) $\rho\left[\dfrac{\partial \vec{V}}{\partial t}-\left(\vec{V}.\nabla\right)\vec{V}\right]=-\nabla p+\rho\vec{b}$

(c) $\rho\left[\dfrac{\partial \vec{V}}{\partial t}+\left(\vec{V}.\nabla\right)\vec{V}\right]=-\nabla p+\rho\vec{b}$

(d) $\rho\left[\dfrac{\partial \vec{V}}{\partial t}-\left(\vec{V}.\nabla\right)\vec{V}\right]=\nabla p+\rho\vec{b}$

[*Ans.(*c)]

(ii) Bernoulli's equation relates

(a) various forces involved in fluid flow
(b) various forms of mechanical energy
(c) torque to change in angular momentum
(d) various forms of chemical energy

[*Ans.(*b)]

(iii) When is Bernoulli's equation applicable between any two points in a flow field?

(a) The flow is steady, constant density and rotational
(b) The flow is unsteady, constant density and irrotational
(c) The flow is steady, variable density and irrotational
(d) The flow is steady, constant density and rotational

[*Ans.(*c)]

(iv) A stagnation point is a point in fluid flow where

(a)enthalpy of fluid is zero
(b) velocity of flow is zero
(c) pressure is zero
(d) internal energy of fluid is zero

[*Ans.(*b)]

Q2.

The velocity components in an inviscid, constant density ($=1000$ kg/m^3), steady flow field are given as follows: $u=\dfrac{A}{2}\left(x+y+z\right)$, $v=\dfrac{A}{2}\left(x+y+z\right)$, $w=-A\left(x+y+z\right)$, where A is a dimensional constant, with a numerical value of 1 unit. Consider a directed line segment in the flow field, connecting the points $\mathrm{P}_1\left(0,0,0\right)$ and $\mathrm{P}_2\left(-3,3,0\right)$. The pressure is given as zero gauge at the origin. Can the Bernoulli's equation be applied to find the change in pressure experienced on moving from the point P_1 to the point P_2 along the direction $\mathrm{P}_1\mathrm{P}_2$?

Solution

Along a streamline

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

or $$\frac{dx}{\frac{A}{2}(x+y+z)}=\frac{dy}{\frac{A}{2}(x+y+z)}=\frac{dz}{-A(x+y+z)}$$

or $$\frac{dy}{dx}=1 \text{ and } \frac{dz}{dy}=-2$$

Again along $\overrightarrow{P_1P_2}$ $$\frac{dy}{dx}=\frac{3-0}{-3-0}=-1\neq 1$$

Therefore, $\overrightarrow{P_1P_2}$ is not parallel to the streamline.

Euler equation along x-direction :

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=\rho b_x-\frac{\partial p}{\partial x}$$

or $$\rho\left[\frac{\partial u}{\partial t}+\left(\vec{V}.\nabla\right)u\right]=-\frac{\partial p}{\partial x}+\rho b_x$$

or $$\frac{\partial p}{\partial x}=-\rho\left[\frac{\partial u}{\partial t}+\left(\vec{V}.\nabla\right)u\right]+\rho b_x$$

Similarly, $$\frac{\partial p}{\partial y}=-\rho\left\{\frac{\partial v}{\partial t}+\vec{V}.\nabla v\right\}+\rho b_y$$

$$\frac{\partial p}{\partial z}=-\rho\left\{\frac{\partial w}{\partial t}+\vec{V}.\nabla w\right\}+\rho b_z$$

Pressure difference between any two points 1 and 2 can be written as

$$dp=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy+\frac{\partial p}{\partial z}dz$$

Putting the expressions of $\frac{\partial p}{\partial x}$, $\frac{\partial p}{\partial y}$, and $\frac{\partial p}{\partial z}$, we get

$$dp=\underbrace{-\rho\left[\frac{\partial u}{\partial t}dx+\frac{\partial v}{\partial t}dy+\frac{\partial w}{\partial t}dz\right]}_{\text{Term 1}}\underbrace{-\rho\left[\left(\vec{V}.\nabla u\right)dx+\left(\vec{V}.\nabla v\right)dy+\left(\vec{V}.\nabla w\right)dz\right]}_{\text{Term 2}}+\underbrace{\rho\left[b_xdx+b_ydy+b_zdz\right]}_{\text{Term 3}}$$

Term 1 $$=-\rho\left[\frac{\partial u}{\partial t}dx+\frac{\partial v}{\partial t}dy+\frac{\partial w}{\partial t}dz\right]=-\rho\left[\frac{\partial \vec{V}}{\partial t}.d\vec{l}\right]$$

where $d\vec{l}$ directed from point 1 to 2 be such that

$$d\vec{l}=dx\hat{i}+dy\hat{j}+dz\hat{k}$$

For simplifying Term 2, we first note the vector identity

$$\left(\vec{V}.\vec{\nabla}\right)\vec{V}=\frac{1}{2}\nabla\left(\vec{V}.\vec{V}\right)-\vec{V}\times\left(\nabla\times\vec{V}\right)$$

$$=\frac{1}{2}\nabla\left(\vec{V}.\vec{V}\right)-\vec{V}\times\vec{\xi}$$ (noting that the vorticity, $\vec{\xi}=\nabla\times\vec{V}$)

Term 2 $= -\rho\left[\left(\vec{V}.\nabla u\right)dx + \left(\vec{V}.\nabla v\right)dy + \left(\vec{V}.\nabla w\right)dz\right] = -\rho\left[\left(\vec{V}.\vec{\nabla}\right)\vec{V}.d\vec{l}\right]$

$= -\rho\left[\frac{1}{2}\nabla\left(\vec{V}.\vec{V}\right) - \vec{V}\times\vec{\xi}\right].d\vec{l}$ (using the above vector identity)

$$= -\rho\left[\frac{1}{2}\left\{\hat{i}\frac{\partial V^2}{\partial x} + \hat{j}\frac{\partial V^2}{\partial y} + \hat{k}\frac{\partial V^2}{\partial z}\right\}.d\vec{l} - \left\{\vec{V}\times\vec{\xi}\right\}.d\vec{l}\right]$$

$$= -\rho\left[\frac{1}{2}\left\{\hat{i}\frac{\partial V^2}{\partial x} + \hat{j}\frac{\partial V^2}{\partial y} + \hat{k}\frac{\partial V^2}{\partial z}\right\}.(\hat{i}dx + \hat{j}dy + \hat{k}dz) - \left\{\vec{V}\times\vec{\xi}\right\}.d\vec{l}\right]$$

$$= -\rho\left[\frac{1}{2}\left\{\frac{\partial}{\partial x}\left(V^2\right)dx + \frac{\partial}{\partial y}\left(V^2\right)dy + \frac{\partial}{\partial z}\left(V^2\right)dz\right\}\right] + \rho\left[\left(\vec{V}\times\vec{\xi}\right).d\vec{l}\right]$$

$$= -\frac{1}{2}\rho\left(dV^2\right) + \rho\left[\left(\vec{V}\times\vec{\xi}\right).d\vec{l}\right]$$

$$\vec{\xi} = \nabla\times\vec{V} = \begin{vmatrix} \hat{i} & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{A}{2}(x+y+z) & \frac{A}{2}(x+y+z) & -A(x+y+z) \end{vmatrix}$$

$$= \hat{i}\left(-A - \frac{A}{2}\right) - \hat{j}\left(-A - \frac{A}{2}\right) + \hat{k}\left(\frac{A}{2} - \frac{A}{2}\right)$$

$$= -\frac{3A}{2}\hat{i} + \frac{3A}{2}\hat{j}$$

$$\vec{V}\times\vec{\xi} = \begin{vmatrix} \hat{i} & j & k \\ \frac{A}{2}(x+y+z) & \frac{A}{2}(x+y+z) & -A(x+y+z) \\ -\frac{3A}{2} & \frac{3A}{2} & 0 \end{vmatrix}$$

$$= \hat{i}\left[-\frac{3A^2}{2}(x+y+z)\right] - \hat{j}\left[\frac{3A^2}{2}(x+y+z)\right] + \hat{k}\left[-\frac{3A^2}{4}(x+y+z) - \frac{3A^2}{4}(x+y+z)\right]$$

$$= \left(-\frac{3}{2}\hat{i} - \frac{3}{2}\hat{j} - \frac{9}{2}\hat{k}\right)\frac{A^2}{2}(x+y+z)$$

$$d\vec{l} = -3\hat{i} + 3\hat{j}$$

$$\left(\vec{V}\times\vec{\xi}\right).d\vec{l} = \frac{9}{2} - \frac{9}{2} = 0$$

Since, $\left(\vec{V}\times\vec{\xi}\right).d\vec{l} = 0$, Bernoulli's equation can be applied.

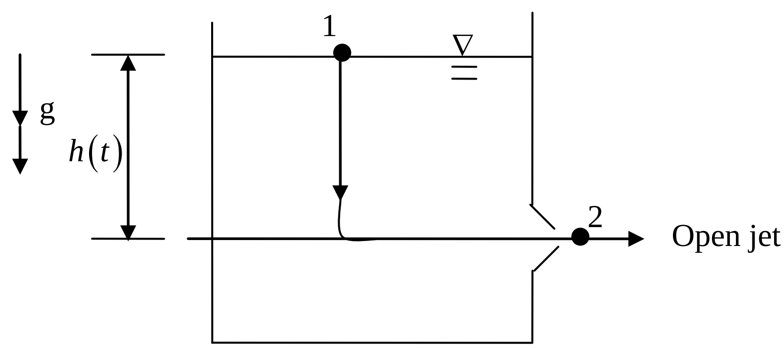
Term 3 $\rho\left[b_x dx + b_y dy + b_z dz\right] = \rho\vec{b}.d\vec{l}$

$$dp = -\frac{1}{2}\rho\left(dV^2\right) + \rho\vec{b}.d\vec{l}$$

Integrating the above equation with constant ρ, one can get Bernoulli's equation.

Q3.

Water comes out of a tank through a nozzle, as an open jet. As a result, the level of water in the tank continuously falls. A streamline in the tank conceptually identifies in the figure below (Spanning from point 1 to point 2), the curvilinear length of which (spanning from the point 1 to the point 2) is approximately kh, where $k = 1.5$. For mathematical analysis, following assumptions can be made: (i) velocity of flow along the streamline is approximately V_1, and (ii) viscous effects are negligible. The ratio of area of cross section of the tank to that of the nozzle is 2: 1. At a given instant of time, $h =$ 5 m and $V_2 = 1$ m/s. What is the local component of acceleration of flow at the point 2, at that instant?



Solution

Applying unsteady Bernoulli's equation along the streamline between points 1 and 2, we have

$$\frac{p_2 - p_1}{\rho} + \frac{V_2^2 - V_1^2}{2} + g\left(z_2 - z_1\right) + \int_1^2 \frac{\partial V}{\partial t} ds = 0$$

Here $\quad p_2 = p_1 = p_{atm}$

$$\int_1^2 \frac{\partial V}{\partial t} ds \approx \int_1^2 \frac{\partial V_1}{\partial t} ds \approx \frac{\partial V_1}{\partial t} \int_1^2 ds = kh\frac{\partial V_1}{\partial t}$$

$$z_2 - z_1 = -h$$

From continuity equation,

$$A_1 V_1 = A_2 V_2$$

or

$$V_1 = \frac{A_2}{A_1} V_2$$

Thus

$$\frac{V_2^2}{2}\left[1 - \left(\frac{A_2}{A_1}\right)^2\right] - gh + kh\frac{\partial V_1}{\partial t} = 0$$

or $$\frac{\partial V_1}{\partial t} = \frac{g - \frac{V_2^2}{2h}\left[1-\left(\frac{A_2}{A_1}\right)^2\right]}{k}$$

Substituting the values, we obtain

$$\frac{\partial V_1}{\partial t} = \frac{9.81 - \frac{1^2}{2\times 5}\left[1-\left(\frac{1}{2}\right)^2\right]}{1.5} = 6.49 \text{ m/s}^2$$

Q4.

A vertical venturimeter carries a liquid of specific gravity 0.8 and has an inlet and throat diameter of 150 mm and 75 mm, respectively. The pressure connection at the throat is 150 mm above that at the inlet. If the actual rate of flow is 40 litres/s and the coefficient of discharge is 0.96, calculate (i) the pressure difference between inlet and throat, and (ii) the difference in levels of mercury in a vertical U-tube manometer connected between these points.

Solution

(i)Let A and B represent the inlet and the throat sections respectively.

Then $$V_A = \left(\frac{0.075}{0.15}\right)^2 V_B = \frac{V_B}{4}$$

Applying Bernoulli's equation between A and B, along a streamline, we get

$$\frac{p_A}{\rho g} + \frac{V_A^{\,2}}{2g} + z_A = \frac{p_B}{\rho g} + \frac{V_B^{\,2}}{2g} + z_B$$

or $$\frac{V_B^{\,2} - V_A^{\,2}}{2g} = \frac{p_A - p_B}{\rho g} - 0.15 \qquad \left[\because z_B - z_A = 0.15 \text{ m}\right]$$

or $$\frac{V_B^{\,2} - \frac{V_B^{\,2}}{16}}{2g} = \frac{p_A - p_B}{\rho g} - 0.15$$

or $$\frac{15 V_B^{\,2}}{32g} = \frac{p_A - p_B}{\rho g} - 0.15$$

or $$V_B = \sqrt{\frac{32g}{15}\left(\frac{p_A - p_B}{\rho g} - 0.15\right)}$$

The actual rate of discharge Q_{actual} can be written as

$$Q_{actual} = C_d \times V_B \times \text{ area of the throat}$$

or $$0.4 = 0.96 \times \sqrt{\frac{32g}{15}\left(\frac{p_A - p_B}{\rho g} - 0.15\right)} \times \pi \frac{(0.075)^2}{4}$$

which gives

$$p_A - p_B = 33.82 \text{ kN/m}^2$$

(b)

Let *h* be the difference in the level of mercury in the manometer.

Then,

$$\frac{p_A - p_B}{\rho g} = h\left(\frac{13.6}{0.8} - 1\right)$$

or $$33.82 \times 10^3 = 1000 \times 0.8 \times 9.81 \times 16h$$

which gives $$h = 0.26 \text{ m}$$