Kinematics of Fluid (Lectures 11 to 15)

Q1. Choose the correct answer

(i) The streamline shapes of the following 2-D velocity field: u = −y, v = x will be

(a) circle
(b) parabola
(c) ellipse
(d) rectangular hyperbola

[*Ans.(*d)]

(ii) Fluid flows steadily through a converging nozzle of length *L*. Flow can be approximated as one-dimensional such that the axial velocity varies linearly from entrance to exit. The velocities at entrance and exit are V_0 and $4V_0$ respectively. The acceleration of a particle flowing through the nozzle is given by

(a) $\frac{V_0^2}{L}\left(1+\frac{3x}{L}\right)$

(b) $\frac{2V_0^2}{L}\left(1+\frac{3x}{L}\right)$

(c) $\frac{3V_0^2}{L}\left(1+\frac{3x}{L}\right)$

(d) $\frac{4V_0^2}{L}\left(1+\frac{3x}{L}\right)$

[*Ans.(*c)]

(iii) Continuity equation for a given velocity filed is given by $\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0$.

The above equation is applicable only for

(a) incompressible flow.
(b)inviscid flow
(c)steady and 2-dimensional flow
(d)two-dimensional and incompressible flow

[*Ans.(*c)]

(iv) A two dimensional flow in x-y plane is irrotational if

(a) $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$

(b) $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}$

(c) $\frac{\partial v}{\partial x}=\frac{\partial u}{\partial y}$

(d) $\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}$

[*Ans.(*c)]

(v) For a two-dimensional irrotational flow, the velocity potential is defined as $\phi = x^2 - y^2$. What will be the stream function (ψ) with the condition $\psi = 0$ at $x = y = 0$?

(a) $x^2 + y^2$

(b) $x^2 - y^2$

(c) $2xy$

(d) $2x^2 y^2$

[*Ans.(*b)]

Q2.

The velocity components in a flow-field are given as follows: $u = x(1+2t)$, $v = y$, $w = 0$. A coloured dye is injected at the point A (1,1) in the flow-field at $t = 0$.

(a) Find the equation of a coloured line visible in the flow-field at $t = 1$, as a consequence of the dye injection, given that the dye injection starts at $t = 0$.

(b) Find the locus of a fluid particle that passes through the point A at $t = 1$.

Solution

(a) Here, essentially, we need to obtain the equation of streakline at $t = 1$. In order to obtain the same, we consider that fluid particles are injected through the point at instants of time symbolically denoted by t_i, where $t_i \le 1$. Importantly t_i is a variable and not a fixed instant of time. Now, based on the given velocity field, we have

$$u = \frac{dx}{dt} = x(1+2t)$$

$$\int_1^x \frac{dx}{x} = \int_{t_i}^1 (1+2t)\, dt$$

$$\ln \frac{x}{1} = \left[t + t^2\right]_{t_i}^1$$

$$\ln x = 2 - t_i - t_i^2 \qquad (1)$$

Similarly,

$$v = \frac{dy}{dt} = y$$

$$\int_1^y \frac{dy}{y} = \int_{t_i}^1 dt$$

$$\ln \frac{y}{1} = [t]_{t_i}^1$$

$$\ln y = 1 - t_i \qquad (2)$$

Eliminating t_i from Eqs (1) and (2), we have

$$\ln x = 2 - (1 - \ln y) - (1 - \ln y)^2$$

$$= 2 - 1 + \ln y - 1 + 2\ln y - (\ln y)^2 = 3\ln y - (\ln y)^2$$

or

$$(\ln y)^2 - 3\ln y + \ln x = 0 \qquad (3)$$

Equation (3) is the equation of streakline passing through the point A at $t = 1$.

(b)Here, essentially, we need to obtain the equation of the pathline, corresponding to a fluid particle that at $t=1$ passed through $(1,1)$

Based on the given velocity field, we have

$$u=\frac{dx}{dt}=x(1+2t)$$

$$\int_1^x \frac{dx}{x}=\int_0^t (1+2t)\,dt$$

$$\ln\frac{x}{1}=\left[t+t^2\right]_0^t$$

$$\ln x=t+t^2 \qquad (4)$$

Similarly, for the *y*-component of velocity, we have

$$v=\frac{dy}{dt}=y$$

$$\int_1^y \frac{dy}{y}=\int_0^t dt$$

$$\ln\frac{y}{1}=\left[t\right]_0^t$$

$$\ln y=t \qquad (5)$$

Eliminating t from Eqs (4) and (5), we have

$$\ln x=\ln y+(\ln y)^2 \qquad (6)$$

Equation (6) is the equation of a pathline in the flow-field at $t=1$.

Q3.

For a steady two-dimensional incompressible flow through a nozzle, the velocity field is given by $\vec{V}=u_0\left(1+\frac{2x}{L}\right)\hat{i}$, where x is the distance along the axis of the nozzle from the inlet plane and L is the length of the nozzle. Find

(i) an expression of the acceleration of a particle flowing through the nozzle and

(ii) the time required for a fluid particle to travel from the inlet to the exit of the nozzle.

Solution

(i)The velocity component is given as

$$u=u_0(1+2x/L)$$

Hence,
$$\frac{\partial u}{\partial x}=\frac{2u_0}{L}$$

For the given velocity field, acceleration can be written as

$$a=u\frac{\partial u}{\partial x}$$

$$a=u_0(1+2x/L)\frac{2u_0}{L}=\frac{2u_0^2}{L}(1+2x/L)$$

(ii)

Let *t* be the time required for a fluid particle to travel from the inlet to the exit of the nozzle.

From the given velocity field, we have

$$V=\frac{dx}{dt}=u_0\left(1+2x/L\right)$$

or
$$\frac{dx}{\left(1+2x/L\right)}=u_0 dt$$

Integrating the above equation, we obtain

$$\int_0^L \frac{dx}{\left(1+2x/L\right)}=\int_0^t u_0 dt$$

which gives
$$t=\frac{L}{2u_0}\ln 3$$

Q4.

Consider the flow field given in Eulerian description by the expression $\vec{V}=\left(x+y+z\right)\hat{i}-\left(xy+yz+zx\right)\hat{j}+\left(w\right)\hat{k}$. The z - component of velocity (w) for which the above flow field represents an incompressible flow is

Solution

For an incompressible flow the continuity equation can be written in differential form as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

Given $u=x+y+z$, and $v=-xy-yz-zx$

Hence, $\frac{\partial u}{\partial x}=1$, and

$$\frac{\partial v}{\partial y}=-x-z$$

Substituting $\frac{\partial u}{\partial x}$, and $\frac{\partial v}{\partial y}$ in the continuity equation, we get

$$1-x-z+\frac{\partial w}{\partial z}=0$$

or,
$$\frac{\partial w}{\partial z}=x+z-1$$

Integrating with respect to *z*, we have

$$w=xz+\frac{z^2}{2}-z+f\left(x,y\right)$$

Q5.

The density field of a steady flow is given by $\rho\left(x,y\right)=Kxy$, where K is a constant. Determine the equation of streamline for which the flow is incompressible

Solution

Given that $\rho\left(x,y\right)=Kxy$

Hence, $\frac{\partial \rho}{\partial t}=0$, $\frac{\partial \rho}{\partial x}=Ky$, $\frac{\partial \rho}{\partial y}=Kx$

For a two-dimensional, steady, incompressible flow the continuity equation can be written in differential form as

$$u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}=0$$

Substituting the values of $\frac{\partial \rho}{\partial x}$ and $\frac{\partial \rho}{\partial y}$ in the continuity equation, we have

$$uKy+vKx=0$$

or $$uy=-vx$$

or $$\frac{v}{u}=-\frac{y}{x}$$

The equation of a streamline in two-dimensional flow is

$$\frac{dx}{u}=\frac{dy}{v}$$

or $$\frac{dy}{dx}=\frac{v}{u}$$

or $$\frac{dy}{dx}=-\frac{y}{x} \qquad (\because \frac{v}{u}=-\frac{y}{x})$$

or $$\frac{dy}{y}=-\frac{dx}{x}$$

Integrating the above equation, we get

$$\ln y=-\ln x+\ln C$$

where C is integration constant

$$\ln xy=\ln C$$

or, $$xy=\text{constant}$$

Required streamline equation is $xy=\text{constant}$

Q6.

A two-dimensional velocity field is given by

$$\vec{V}=-Ay\hat{i}+Ax\hat{j}$$

whre A is a dimensional constant. Find the components of

(a) the strain rates for the above velocity field.

(b) rotational velocity

(c) What will be the physical nature of deformation of a small rectangular fluid element located within this flow-field?

Solution

Given $u=-Ay$ and $v=Ax$

Hence, $\frac{\partial u}{\partial x}=0$, $\frac{\partial u}{\partial y}=-A$, $\frac{\partial v}{\partial x}=A$ and $\frac{\partial v}{\partial y}=0$

(a) Rate of linear strain along x direction is

$$\dot{\varepsilon}_x=\frac{\partial u}{\partial x}=0$$

Rate of linear strain along y direction is

$$\dot{\varepsilon}_y=\frac{\partial v}{\partial y}=0$$

Rate of volumetric strain is

$$\dot{\varepsilon}_{\text{vol}} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 + 0 = 0$$

Rate of shear strain is

$$\dot{\varepsilon}_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = A - A = 0$$

(b) Rotation is given by

$$\omega_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left[A - (-A)\right] = A$$

(c) The physical nature of deformation of a small rectangular fluid element, ABCD located in the above flow-field is schematically shown in the figure below. The given velocity field represents rotation without angular deformation. However, no linear deformation takes place in both *x* and *y* directions. This situation is known as pure rotation without linear and shear deformations.

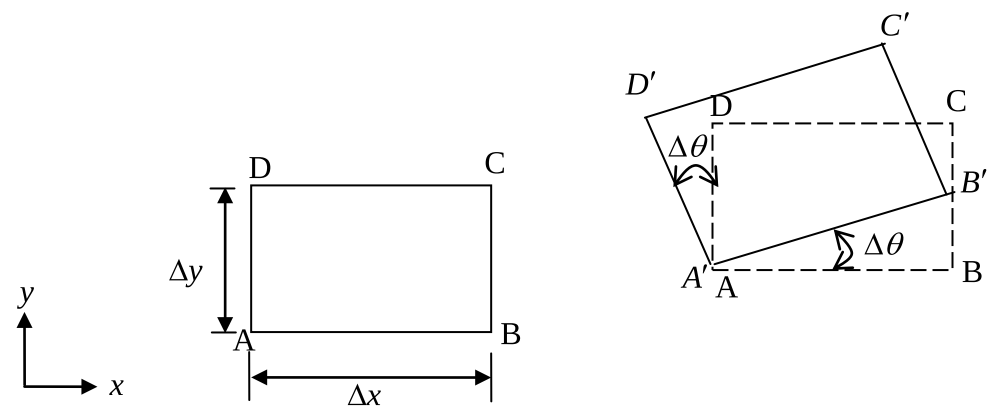

Fig. pure rotation of a fluid element without linear and shear deformations

Q7.

A three-dimensional velocity filed is given by

$$u(x, y, z) = cx + 2w_0 y + u_0$$

$$v(x, y, z) = cy + v_0$$

$$w(x, y, z) = -2cz + w_0$$

where c, w_0, u_0 and v_0 are constants. Find the components of (i) rotational velocity, (ii) vorticity and (iii) the strain rates for the above field.

Solution

(a) The components of the rotational velocity are as follows

$$\dot{\omega}_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(0 - 2w_0\right) = -w_0$$

$$\dot{\omega}_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \frac{1}{2}\left(0 - 0\right) = 0$$

$$\dot{\omega}_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{1}{2}\left(0 - 0\right) = 0$$

(b) The components of the vorticity are as follows

$$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 - 2w_0 = -2w_0$$

$$\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0 - 0 = 0$$

$$\Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 - 0 = 0$$

(c) Rate of linear strain along x direction is

$$\dot{\varepsilon}_{xx} = \frac{\partial u}{\partial x} = c$$

Rate of linear strain along y direction is

$$\dot{\varepsilon}_{yy} = \frac{\partial v}{\partial y} = c$$

Rate of linear strain along x direction is

$$\dot{\varepsilon}_{zz} = \frac{\partial w}{\partial z} = -2c$$

The shear strain rates are found to be

$$\dot{\gamma}_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 + 2w_0 = 2w_0$$

$$\dot{\gamma}_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 + 0 = 0$$

$$\dot{\gamma}_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0 + 0 = 0$$

Q8.

In a two dimensional flow, the fluid velocity components are given by $u = x - 4y$ and $v = -y - 4x$. Check whether the flow is (a) compressible or incompressible, and (b) rotational or irrotational. If the flow is incompressible find the stream function. Find the velocity potential, if the flow is irrotational and vorticity, if it is rotational.

Solution

(a) It is given that

$$u = x - 4y,\ v = -y - 4x$$

Thus, $$\frac{\partial u}{\partial x} = 1 \text{ and } \frac{\partial v}{\partial y} = -1$$

Therefore, $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

The above velocity field satisfies the continuity equation for incompressible flow. Thus the flow is incompressible.

(b)Hence, $\frac{\partial v}{\partial x}=-4$

$$\frac{\partial u}{\partial y}=-4$$

The rotation is given by

$$\omega_z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=\frac{1}{2}(-4+4)=0$$

Since the rotation is zero, the flow is irrotational and hence the velocity potential exists. From the definition of stream function ψ, we get

$$u=\frac{\partial \psi}{\partial y}$$

or $\psi=\int u dy=\int (x-4y)dy$

or $\psi=xy-2y^2+f(x)+C_1$

$$v=-\frac{\partial \psi}{\partial x}$$

$$\psi=-\int v dx=-\int(-y-4x)dx=\int(y+4x)dx$$

or $\psi=xy+2x^2+g(y)+C_2$

Comparing the above two equations, we have

$$f(x)=2x^2,\ g(y)=-2y^2$$

Hence, the stream function for the flow is

$$\psi=xy+2x^2-2y^2+C$$

where, C is a constant.

For irrotational flow, the velocity potential (ϕ) is defined as

$$u=\frac{\partial \phi}{\partial x}$$

$$v=\frac{\partial \phi}{\partial y}$$

Thus, $u=\frac{\partial \phi}{\partial x}=x-4y$

or $\phi=\frac{x^2}{2}-4xy+f_1(y)+C_1$

And $v=\frac{\partial \phi}{\partial y}=-y-4x$

or $\phi=-\frac{y^2}{2}-4xy+f_2(x)+C_2$

Comparing the above two equations, we have

$$f_1(y) = -\frac{y^2}{2}\,,\;\; f_2(x) = \frac{x^2}{2}$$

Hence, the velocity potential for the flow is

$$\phi = \frac{x^2 - y^2}{2} - 4xy + C$$

where, C is a constant.

Q9.

If stream function for steady flow is given by $\psi = y^2 - x^2$, determine whether the flow is rotational or irrotational. Find the potential function, if the flow is irrotational and vorticity, if it is rotational.

Solution

From the definition of stream function ψ, we get

$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x}$$

Thus, the velocity components become

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(y^2 - x^2\right) = 2y$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left(y^2 - x^2\right) = 2x$$

The velocity is then $\vec{V} = u\hat{i} + v\hat{j} = 2y\hat{i} + 2x\hat{j}$

Hence, $\frac{\partial v}{\partial x} = 2$ and $\frac{\partial u}{\partial y} = 2$

The rotation is given by

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(2 - 2\right) = 0$$

Since the rotation is zero, the flow is irrotational and hence the velocity potential exists. For irrotational flow, the velocity potential (ϕ) is defined as

$$u = \frac{\partial \phi}{\partial x}$$

$$v = \frac{\partial \phi}{\partial y}$$

Thus, $u = \frac{\partial \phi}{\partial x} = 2y$

or $\phi = 2xy + f_1\left(y\right) + C_1$ (1)

And $v = \frac{\partial \phi}{\partial y} = 2x$

or, $$\phi = 2xy + f_2(x) + C_2 \quad (2)$$

Comparing Eqs (1) and (2), we have

$$f_1(y) = 0, \quad f_2(x) = 0$$

Hence, the velocity potential for the flow is

$$\phi = 2xy + C$$

where, C is a constant.