# Wave Propagation in Continuous Media

Exercises

1. Find the general solution in Cartesian coordinates to the 1st-order PDE

$$yu_x - xu_y = x^2 + y^2.$$

Verify the solution by direct differentiation.

2. Solve the following equations :

(a) $(y+u)u_x + (x+u)u_y = x + y$.

(b) $xu(u^2 + xy)u_x - yu(u^2 + xy)u_y = x^4$.

3. Determine the solution $u = u(x, y)$ to the PDE

$$uu_x - uu_y = y - x$$

that satisfies $u(1, y) = g(y)$ for a given function $g(y)$.

4. (a) Find the general solution to the PDE

$$yu_x + xu_y = 2xy.$$

(b) Find the particular solution that satisfies $u = y$ on the circle $x^2 + y^2 = 1$.

5. Show that the solution of the nonlinear PDE $u_x + u_y = u^2$ that satisfies $u = x$ on the line $y = -x$ becomes infinite along the hyperbola $x^2 - y^2 = 4$.

6. Find the general solution of the equation $yu_x - 2xyu_y = 2xu$. In particular, determine the solution that satisfies $u(0, y) = y^3$.

7. It may be assumed that the rate of deposit or removal of sand on the bed of a stream is $a(\partial v/\partial x)$, where $a$ is a constant and $v$ is the velocity of the water in the $x$ direction. If $\eta$, $h$ denote the heights, above an arbitrary zero level, of the top of the sand in the bed and of the water surface, respectively, show that the variation of $\eta$ is governed by the first order equation

$$(h-\eta)^2 \frac{\partial \eta}{\partial t} + m \frac{\partial \eta}{\partial x} = 0$$

where $m$ is a constant. Assuming $h$ to be constant, show that the general solution of this equation is

$$\eta = f[x - mt/(h-\eta)^2]$$

where the function $f$ is arbitrary. If $\eta = \eta_0 \cos(2\pi x/\lambda)$ at $t = 0$, find the relation between $\eta$ and $x$ at time $t$.

8. Consider the quasi-linear PDE

$$uu_x + u_y = 0, \qquad -\infty < x < \infty, \quad y > 0,$$

with the initial data $u(x,0) = \operatorname{sech} x$, $-\infty < x < \infty$. Determine the solution, and show that the solution remains differentiable so long as $y < y^*$, where $y^*$ is the minimum value of the function

$$\frac{\cosh^2 s}{\sinh s}, \quad s > 0.$$

9. Use characteristics in order to solve the system of PDE

$$u_t + v_x = 0, \quad v_t + u_x = 0$$

with the initial data $u(x,0) = f(x)$ and $v(x,0) = g(x)$.

10. Determine the dispersion relation for the wave equation with a mixed derivative, $u_{tt} - c^2 u_{xx} = -2\alpha u_{xt}$, where $\alpha$ and $c$ are positive constants. Plot the dispersion relation in the $\omega$-$k$ plane. Also, plot the variation of the phase and group velocities with $k$.

11. (a) Find the general solution $u(x,y)$ of the PDE

$$u_x + 2x(e^{-x^2} - y)u_y = x^2.$$

(b) Describe (with the help of a sketch) the region R of the xy-plane in which $u(x,y)$ is determined by prescribed values of $u$ along the line segment that connects the points (0,0) and (0,1). If $u = 1 + y$ on this line segment, find $u(x,y)$ in $R$.

12. Determine the solution $u = u(x,y,z)$ of the PDE $xu_x + yu_y + uu_z = 0$, where u satisfies the condition $u(x,y,0) = xy$ for $x > 0, y > 0$.

13. Using the Fourier and Laplace transforms, solve the heat equation $w_{,t} = \alpha w_{,xx}$, $x \in (-\infty, \infty)$ with the initial conditions $w(x,0) = \delta(x)$. Using this solution, write the solution for a general initial condition $w(x,0) = g(x)$.

14. Determine the Green's funtion of an infinite string, i.e., the solution of $w_{,tt} - c^2 w_{,xx} = \delta(x)\delta(t)$ with $w(x, 0) = 0$ and $w_{,t}(x, 0) = 0$. Using this solution, express the solution of the system $w_{,tt} - c^2 w_{,xx} = 0$ with $w(x, 0) = w_0(x)$ and $w_{,t}(x, 0) = v_0(x)$.

15. Flexural vibrations of a beam is governed by $w_{,tt} + \beta^4 w_{,xxxx} = 0$. Write down the dispersion relation and plot. Also calculate the phase and group velocities. Determine the exact evolution of a Gaussian wave packet in a beam.

16. A string is actuated tranversely by an actuator at one end while the other end is connected to a viscous damper of damping coefficient $d$. Both ends can slide transversely. Determine the impedance of the system as observed by the actuator.

17. A homogeneous uniform bar, fixed at one end, is kept under axial tension by a string. If the string suddenly snaps, determine the motion of the bar in terms of the propagating waves. Take $\rho$, $A$, $E$ and $l$ as, respectively, the density, area of cross-section, Young's modulus and length of the bar.

18. A particular type of electromagnetic waves in a dissipative medium are governed by the PDE $u_{tt} - c^2 u_{xx} + 2\mu(u_t + a u_x) = 0$, where $c > 0, \mu > 0$, and $a$ is real.

    (a) Find the relation $\omega = W(k)$ for a plane wave $u = e^{i(kx - \omega t)}$ in this medium. Calculate the phase and group velocities as a functions of $k$.

    (b) Derive approximations to $W(k)$ for fixed $k$ in the following cases: (i) For sufficiently small $\mu$, retain terms to order $\mu$ and show that $u$ decays if $c > a > -c$.

    (ii) For sufficiently large $\mu$, retain leading terms and show that no waves can grow in time. Show that this case is equivalent to the approximation of small $k$ (i.e., long waves) for fixed $\mu$.