Module-1: Tensor Algebra

Lecture-1: Introduction to Continuum Mechanics

Continuum mechanics is a branch of mathematical physics and it deals with the deformation of matter under the action of forces and thermal effects. The treatment is given for all the forms of matter (solids, liquids, and gasses) in a unified framework. The framework of continuum mechanics is developed by assuming fundamental laws of mechanics and thermodynamics as axioms.

Structure of matter

Well known fact from elementary physics that the matter is composed of molecules, which in turn consists of atoms. Further, the atoms are composition of elementary particles (electrons, protons, neutrons, etc.). Thus, the matter is composition of elementary particles irrespective of the phase of the matter (i.e., solid, liquid or gas state). Consequently, the material properties of matter are related directly to the molecular structure and also to the intermolecular forces. These forces not only depends on structure of molecules but also on intermolecular distances. Although the molecules in the matter always undergo random vibration/motion, the bulk material exhibits a stable behavior at macroscopic level. Hence, the study of deformation behavior of matter can be approached fundamentally in two ways: (i) considering the molecules as discrete particles which is known as *statistical mechanics* (ii) considering the bulk material as continuous medium which is known as *continuum mechanics*.

Statistical mechanics

This approach seems useful in unifying the mechanics as every material in any phase consists of atoms. In this approach, we need to know the intermolecular potentials to predict the behavior of matter. It can be noted that the separation distance between two adjacent atoms (i.e., center to center distance) is approximately in the order of 10^{-8} cm. Furthermore, there are about 10^{22} atoms present in a gram of copper. Therefore, this approach becomes too complicated to predict the desired results in any large scale problem due to the computational difficulties that arise in evaluating particle interactions. Apart from the computational complexity, it is also very difficult to know the exact intermolecular potentials. Also, small error in the intermolecular potentials can lead to a large error in predicted results. Hence, this approach has limited usage for large scale problems such as failure of turbine blade, building response under seismic waves and drag forces on aircraft. Thus, the continuum mechanics approach is preferred over the statistical mechanics in order to solve large scale problems in physics and engineering.

Continuum assumption

In nature, despite having random vibration/motion of molecules, we observe the definite behavior for the bulk material (i.e., the matter at macroscopic scale). For example there is definite shape for a wooden block and water in rotating bucket assumes paraboloid shape. Therefore, the matter can be treated as continuum (i.e., the matter without tiny holes) at macroscopic scale. In other words, the molecular structure can be disregarded. In fact, historically, due to stable behavior of matter at macro-scale, the study of mechanics of solids and fluids came into existence much before the discovery of molecular and atomic structure of matter.

Continuum mechanics and its objective:
The fluid continuously deform (flow) under action of forces while solids exhibit the finite deformation. This distinct quality of fluids and solids lead to the development of two diverse branches of mechanics called solid mechanics and fluid mechanics. Although the fluids and solids exhibits entirely different behavior, it should be noted that both of them follow conservation of momenta and energy. Hence, we expect common governing laws for both fluids and solids. Furthermore, the continuum mechanics brings common framework based on fundamental laws of mechanics and thermodynamics. Though both fluids and solids possess molecules as building block, the continuum mechanics framework is developed without considering the molecular nature of matter. This unified framework also helps in study of some complex material which exhibit both solid and fluid like behaviors known as viscoelastic solids and viscoelastic fluids.

Comparison of continuum and statistical mechanics:
We develop the concept of stress in continuum mechanics which is equivalent to intermolecular forces in statistical mechanics. We define geometric quantities called strain and strain rate in continuum mechanics and they are equivalent to intermolecular distance in statistical mechanics. The constitutive relations in continuum mechanics assume the role of intermolecular potentials in statistical mechanics.

Need for tensor analysis
As pointed out in previous discussion, we define two new quantities called stress and strain. These quantities can not be represented either by scalars or vectors. Therefore, a new mathematical object *tensor* is introduced to represent these new physical quantities in continuum mechanics. There are two advantages with the definition of tensor:

- The more general definition of tensor accounts scalars and vectors as special cases. In other words the tensor unifies the definition of physical quantities.
- The physical quantities defined using concept of tensor can be independent coordinate frame. Hence, tensors are ideally suited for describing basic (fundamental) laws of continuum physics.

Continuum nature of real numbers
We need numbers not only for counting but also for quantifying any physical quantities such as density and temperature. We know that the rational numbers are sufficient to quantify all quantities that are encountered in our practice up to a desired accuracy. However, it can be noted that there are gaps (tiny holes) in the rational numbers. In fact, however close may be two distinct rational number, we can always find an irrational number between them. Although the rational numbers serves all the practical purpose

of quantification, there is a difficulty arises in the field of rational numbers while taking limiting process. Hence, it is essential to fill the gaps with irrational numbers. The rational numbers and irrational number put together forms a gap-less number system called *real numbers*. Therefore, the real number system can be thought of as one dimensional continuum since there are no gaps. We take the advantage of the limiting process of real numbers in continuum mechanics.

Three-dimensional Euclidean space and continuum body:
Cartesian product of three sets of real numbers, i.e. the set of all triplets (x, y, z), where x, y and z are real numbers, is sufficient to represent every point in our real world. Furthermore, the Cartesian product of three sets of real numbers is called three-dimensional (3-D) Euclidean space. The 3-D Euclidean space is a continuum as the real number set is continuum. Consequently, it is convenient to represent continuum bodies as bounded domain in the 3-D space.

The concept of continuum or continuous media allow us to define piecewise continuous functions on material domain (continuum body) and also permit us to take limits and differentiation at a point in order to define the stress. Therefore, this approach enables us to use powerful calculus of several variables and thereby facilitate the study of nonuniform distribution of stresses in the material body. The quantitative predictions by continuum theories closely agree with the wide range of experimental observations. Therefore, this powerful theory can be utilized in complicated engineering problems.

Limitations of continuum mechanics:
The assumptions of continuum mechanics do not allow us to create new cracks in solids during the process of deformation. Furthermore, the continuum theory is also not applicable at high altitudes from the earth as presence of air is in the form of rarefied gas (i.e., molecules of air sparsely distributed). However, these cases can be modeled using continuum mechanics in combination with some empirical information.

References

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