

## Module 5

### Numerical Techniques

**Lecture1:** Review of numerical methods for finding roots of characteristics equation: method of bisection, Newton-Raphson's method; and numerical solution of differential equation to obtain the time response: Runga-Kutta method, Wilson- Beta method, FFT analysis

**Lecture2:** Frequency response curves: solution of polynomial equations, solution of set of algebraic equations, Continuation Algorithm

**Lecture 3:** Poincare' section of periodic, quasi-periodic and chaotic responses, Basin of attraction: point to point mapping and cell-to-cell mapping, Lyapunov exponents.

## Module 5 Lecture 1

### Review of numerical methods: Root finding and Solution of Deferential Equations

In this lecture, the basic numerical tools used in the study of nonlinear dynamics of a system will be briefly reviewed. It may be recalled from the derivation of equation of motion in module 2 that it is not always possible to use simple expressions or closed form analytical solutions and hence, one requires numerical operations to solve the problems. Generally following numerical tools are required.

- Numerical differentiation
- Numerical integration
- Finding roots of the algebraic or transcendental equation
- Solving differential equation
- Finding eigenvalues
- Poincare' section
- Lyapunov exponent

For example consider the system whose equation of motion we have derived in module 2.

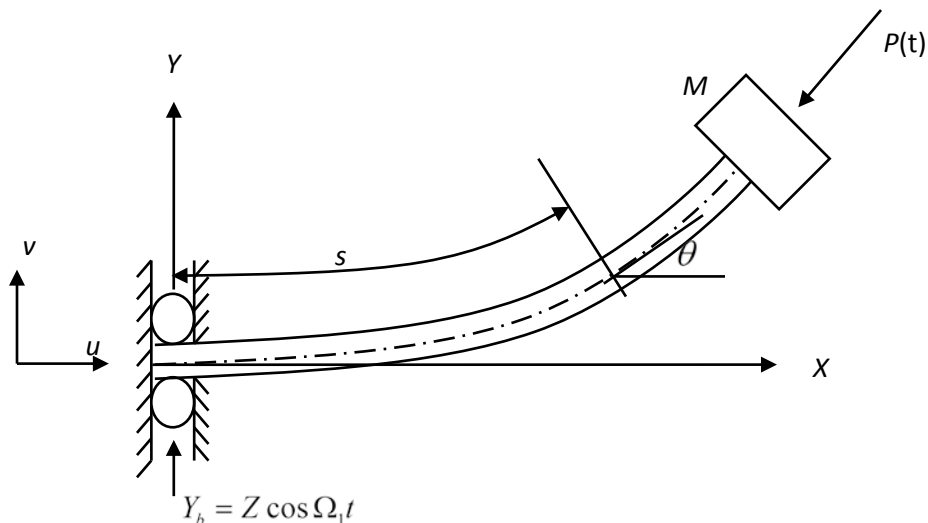


Figure 5.1.1: Schematic diagram of a single-link Cartesian manipulator with payload subjected to harmonically varying axial force.

Figure 5.1.1 shows the system with a payload of mass  $m$  at the tip where a compressive force  $P = P_0 + P_1 \cos \Omega_2 t$  is applied. Also this system is subjected to a harmonic base excitation  $Y_b(t) = Z \cos \Omega_1 t$  at the roller supported left end. Here  $Z$  and  $\Omega_1$  are the amplitude and frequency of the base excitation,  $P_0, P_1$  are the static and dynamic force amplitude, and  $\Omega_2$  is the frequency of

the periodic force acting at the free end of the manipulator. The motion is considered to be in the vertical plane.

Using D'Alembert principle and generalized Galerkin's method the temporal equation of this system can be given by [Pratiher and Dwivedy]

$$\ddot{q} + q + 2\varepsilon\zeta\dot{q} + \varepsilon\left(\alpha_1 q^3 + \alpha_2 q^2 \ddot{q} + \alpha_3 \dot{q}^2 q\right) + \varepsilon\left(\alpha_4 \bar{\omega}_1^2 \cos(\bar{\omega}_1 \tau) q^2 + \alpha_5 \bar{\omega}_1^2 \cos(\bar{\omega}_2 \tau) + \alpha_6 \cos(\bar{\omega}_2 \tau) q\right) = 0 \quad (5.1.1)$$

which is same as the equation(2.6.12) derived in module2. The coefficients used in this equation are described below.

The natural frequency ( $\omega_e$ ) of the lateral vibration of an elastic beam

$$\omega_e = \sqrt{\frac{EI}{\rho AL^4} \left(\frac{h_{14}}{h_2}\right) + \frac{P_0}{\rho AL^2} \left(\frac{h_{21}}{h_2}\right)} = \sqrt{\chi \frac{h_{14}}{h_2} + \bar{P}_0 \left(\frac{h_{21}}{h_2}\right)} \quad (5.1.2)$$

$$\text{Damping ratio } (\zeta) = \frac{C_d}{2\varepsilon\rho A\omega_e}, \quad (5.1.3)$$

Coefficient of the nonlinear geometric term  $q^3$

$$= \alpha_1 = \frac{\chi \bar{\lambda}^2}{\varepsilon \omega_e^2} \left(\frac{h_{19}}{h_2} + \frac{h_{18}}{2h_2} + 3\frac{h_{20}}{h_2}\right) \quad (5.1.4)$$

Coefficient of the nonlinear inertia term  $q^2 \ddot{q}$

$$= \alpha_2 = \frac{\bar{\lambda}^2}{\varepsilon} \left(\frac{h_3}{h_2} + \frac{h_4}{h_2} + \bar{m} \frac{h_5}{h_2} - \frac{h_6}{h_2} - \bar{m} \frac{h_7}{h_2} - \frac{h_8}{h_2}\right), \quad (5.1.5)$$

Coefficient of the nonlinear inertia term  $\dot{q}^2 q$

$$= \alpha_3 = \frac{\bar{\lambda}^2}{\varepsilon} \left(\frac{h_{11}}{h_2} - \frac{h_{12}}{h_2} - \bar{m} \frac{h_{13}}{h_2}\right) \quad (5.1.6)$$

Coefficient of the term  $q^2 \cos(\bar{\omega}_1 \tau)$

$$= \alpha_4 = \frac{\bar{Z}\bar{\lambda}}{\varepsilon} \left(\frac{h_{15}}{h_2} + \bar{m} \frac{h_{16}}{h_2} - \frac{h_{17}}{2h_2}\right) \quad (5.1.7)$$

$$\text{Coefficient of the direct forced term } (\cos(\bar{\omega}_2\tau)), \alpha_5 = \frac{\bar{r}}{\varepsilon} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad (5.1.8)$$

$$\text{Coefficient of the parametric excitation } (q \cos(\bar{\omega}_2\tau)), \alpha_6 = \frac{P_1}{M \omega_e^2 L^2} \begin{pmatrix} h_{21} \\ h_2 \end{pmatrix} = \bar{P}_1 \begin{pmatrix} h_{21} \\ h_2 \end{pmatrix}. \quad (5.1.9)$$

Where

$$\begin{aligned} h_1 &= \int_0^1 \psi_n^{iv} \psi_n ds, \quad h_2 = \int_0^1 [\psi(\bar{x})]^2 d\bar{x}, \quad h_3 = \int_0^1 \left[ \frac{d\psi(\bar{x})}{d\bar{x}} \left( \int_0^x \left[ \frac{d\psi(\bar{\xi})}{d\bar{\xi}} \right]^2 d\bar{\xi} \right) \psi(\bar{x}) \right] d\bar{x}, \\ h_4 &= \int_0^1 \left[ \frac{d\psi(\bar{x})}{d\bar{x}} \frac{d^2\psi(\bar{x})}{d\bar{x}^2} \left[ \int_x^1 \psi(\bar{\xi}) d\bar{\xi} \right] \psi(\bar{x}) \right] d\bar{x}, \quad h_5 = \int_0^1 \left[ \frac{d\psi(\bar{x})}{d\bar{x}} \frac{d^2\psi_y(\bar{x})}{d\bar{x}^2} [\psi(\bar{x})]^2 \right] d\bar{x}, \\ h_6 &= \int_0^1 \left[ \frac{d^2\psi(\bar{x})}{d\bar{x}^2} \left( \int_x^1 \int_0^\eta \left[ \frac{d\psi(\bar{x})}{d\bar{\xi}} \right]^2 d\bar{\xi} d\bar{\eta} \right) \psi(\bar{x}) \right] d\bar{x}, \quad h_7 = \int_0^1 \left[ \frac{d^2\psi(\bar{x})}{d\bar{x}^2} \left( \int_0^x \left[ \frac{d\psi(\bar{\xi})}{d\bar{\xi}} \right]^2 d\bar{\xi} \right) \psi(\bar{x}) \right] d\bar{x}, \\ h_8 &= \int_0^1 \left[ \frac{d\psi(\bar{x})}{d\bar{x}} \right]^2 [\psi(\bar{x})]^2 d\bar{x}, \quad h_9 = \int_0^1 \left[ \frac{d\psi(\bar{x})}{d\bar{x}} \frac{d^2\psi(\bar{x})}{d\bar{x}^2} \left[ \int_x^1 \psi(\bar{\xi}) d\bar{\xi} \right] \psi(\bar{x}) \right] d\bar{x}, \\ h_{10} &= \int_0^1 \left[ \frac{d\psi(\bar{x})}{d\bar{x}} \right]^2 [\psi(\bar{x})]^2 d\bar{x}, \quad h_{11} = \int_0^1 \left[ \frac{d\psi(\bar{x})}{d\bar{x}} \left( \int_0^x \left[ \frac{d\psi(\bar{\xi})}{d\bar{\xi}} \right]^2 d\bar{\xi} \right) \psi(\bar{x}) \right] d\bar{x}, \\ h_{12} &= \int_0^1 \left[ \frac{d^2\psi(\bar{x})}{d\bar{x}^2} \left( \int_x^1 \int_0^\eta \left[ \frac{d\psi(\bar{\xi})}{d\bar{\xi}} \right]^2 d\bar{\xi} d\bar{\eta} \right) \psi(\bar{x}) \right] d\bar{x}, \quad h_{13} = \int_0^1 \left[ \frac{d^2\psi(\bar{x})}{d\bar{x}^2} \left( \int_0^x \left[ \frac{d\psi(\bar{\xi})}{d\bar{\xi}} \right]^2 d\bar{\xi} \right) \psi(\bar{x}) \right] d\bar{x}, \\ h_{14} &= \int_0^1 \left[ \frac{d^4\psi(\bar{x})}{d\bar{x}^4} \psi(\bar{x}) \right] d\bar{x}, \quad h_{15} = \int_0^1 \left[ (1-\bar{x}) \frac{d\psi(\bar{x})}{d\bar{x}} \frac{d^2\psi(\bar{x})}{d\bar{x}^2} \psi(\bar{x}) \right] d\bar{x}, \\ h_{16} &= \int_0^1 \left[ \frac{d\psi(\bar{x})}{d\bar{x}} \frac{d^2\psi(\bar{x})}{d\bar{x}^2} \psi(\bar{x}) \right] d\bar{x}, \quad h_{17} = \int_0^1 \left[ \frac{d\psi(\bar{x})}{d\bar{x}} \right]^2 \psi(\bar{x}) d\bar{x}, \quad h_{18} = \int_0^1 \left[ \left( \frac{d\psi(\bar{x})}{d\bar{x}} \right)^2 \frac{d^4\psi(\bar{x})}{d\bar{x}^4} \psi(\bar{x}) \right] d\bar{x}, \\ h_{19} &= \int_0^1 \left[ \frac{d^2\psi(\bar{x})}{d\bar{x}^2} \psi(\bar{x}) \right] d\bar{x}, \quad h_{20} = \int_0^1 \left[ \frac{d\psi(\bar{x})}{d\bar{x}} \frac{d^2\psi(\bar{x})}{d\bar{x}^2} \frac{d^3\psi(\bar{x})}{d\bar{x}^3} \psi(\bar{x}) \right] d\bar{x}, \end{aligned}$$

$$\text{and } h_{21} = \int_0^1 \left[ \frac{d^2 \psi(\bar{x})}{d\bar{x}^2} \psi(\bar{x}) \right] d\bar{x}. \quad (5.1.10)$$

Here,  $E, I, \rho, A, L$  and  $c_d$  are the Young modulus, moment of inertia, mass density, area of cross-section, length of the cantilever beam and damping factor of the system and  $\zeta, \eta$  respectively, are used as integration variables. The following non-dimensional parameters are used in the further analysis.

$$\bar{x} = \frac{s}{L}, \tau = \omega t, \bar{\omega}_1 = \frac{\Omega_1}{\omega}, \bar{\omega}_2 = \frac{\Omega_2}{\omega}, \bar{\lambda} = \frac{r}{L}, \bar{m} = \frac{M}{\rho AL}, \chi = \frac{EI}{\rho AL^4}, \bar{r} = \frac{Z}{r}, \text{ and } \bar{Z} = \frac{Z}{L}.$$

Also, the eigenfunction for a cantilever beam with a tip mass is given by

$$\psi_i(s) = - \left( \frac{\sin \beta_i L + \sinh \beta_i L}{\cos \beta_i L + \cosh \beta_i L} \right) (\cos \beta_i L \bar{x} - \cosh \beta_i L \bar{x}) + (\sin \beta_i L \bar{x} - \sinh \beta_i L \bar{x}) \quad (5.1.11)$$

One may determine  $\beta L$  from the following equation.

$$1 + \cos \beta L \cosh \beta L + \bar{m} \beta L (\cos \beta L \sinh \beta L - \sin \beta L \cosh \beta L) = 0 \quad (5.1.12)$$

One may note that the coefficients of the temporal equation can be obtained by first finding the roots of the characteristic equation (5.1.12)  $\beta L$  and there by finding the integration  $h_1$  to  $h_{21}$  which also requires numerical differentiation. Following numerical methods may be used for finding the roots of the characteristic equation.

- Interval halving-Method of Bisection
- False position (Regula Falsi)
- Newton's method
- Secant Method
- Muller's method

Let us consider a function  $f(\beta L) = f(x) = 0$ . According to method of bisection one should choose two values of  $x$  for example  $x_1$  and  $x_2$  such that, one of the functional value should be positive and other should be negative. Now the next point is obtain by interval halving, i.e.,  $x_3 = \frac{1}{2}(x_1 + x_2)$ . The root lie between  $x_1$  and  $x_3$  if their functional values are of opposite sign. Otherwise, it will lie between  $x_2$  and  $x_3$ . The next point is obtained by halving the interval and the procedure is repeated till one obtains a root having the functional value within a very close tolerance.

In Newton's method which is also known as Newton-Raphson's method one finds the derivative to obtain the root. As shown in Fig. (5.1.2), the actual solution is  $x_0$ . Choosing  $x_k$  as the guess value of the root

$$f'(x_k) = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}$$

Taking  $f(x_{k+1}) = 0$ ,

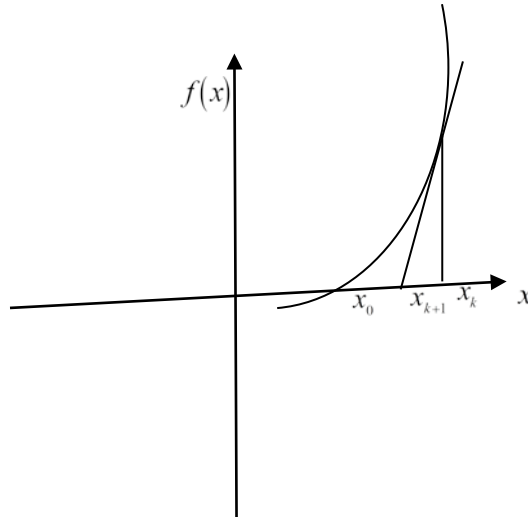


Figure 5.1.2: Illustration of Newton-Raphson method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

It may be noted that point  $x_{k+1}$  is closer to the actual solution  $x_0$ . Now taking  $x_{k+1}$  as the guess value the iteration is repeated till the functional value is within the tolerance limit.

In secant method

In Muller method

For a system of equations, one may use Newton's method.

To solve the differential equations following methods may be used

- The finite difference method,
- The Runge-Kutta method,

- The Wilson method,
- The Newmark method.

### Finite Difference Method

- Forward Difference Method
- Backward difference Method
- Central difference method (Most accurate)

### Central difference method (Most accurate)

- Replace the solution domain with finite number of points (mesh or grid point)
- Using Taylor's series expansion

$$x_{i+1} = x_i + h\dot{x}_i + \frac{h^2}{2}\ddot{x}_i + \frac{h^3}{6}\dddot{x}_i + \dots$$

$$x_{i-1} = x_i - h\dot{x}_i + \frac{h^2}{2}\ddot{x}_i - \frac{h^3}{6}\dddot{x}_i + \dots$$

## RUNGE-KUTTA METHOD

Here the approximate formula used for obtaining  $x_{i+1}$  from  $x_i$  is made to coincide with the

Taylor series expansion of  $x$  at  $x_{i+1}$  up to terms of order  $(\Delta t)^n$ . The Taylor series of expansion of  $x(t)$  at  $t+\Delta t$  is given by-

$$x(t + \Delta t) = x(t) + \dot{x}\Delta t + \ddot{x}\frac{(\Delta t)^2}{2!} + \dddot{x}\frac{(\Delta t)^3}{3!} + \dots$$

### Example:

- For a viscously damped single degree of freedom nonlinear system, we can write

$$\ddot{x} = \frac{1}{m} [F(t) - c\dot{x} - kx - \alpha x^3] = f(x, \dot{x}, t)$$

Taking  $x_1 = x$ , and  $x_2 = \dot{x}$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = f(x_1, x_2, t)$$

The value of  $\bar{X}(t) = [x_1, x_2]'$  at any time  $t$  can be given by using 4<sup>th</sup> order Runge-Kutte method using the following formula.

$$\bar{X}_{i+1} = \bar{X}_i + \frac{1}{6} [\bar{K}_1 + 2\bar{K}_2 + 2\bar{K}_3 + \bar{K}_4]$$

$$\bar{K}_1 = h\bar{F}(\bar{X}_i, t_i)$$

$$\bar{K}_2 = h\bar{F}\left(\bar{X}_i + \frac{1}{2}\bar{K}_1, t_i + \frac{1}{2}h\right)$$

$$\bar{K}_3 = h\bar{F}\left(\bar{X}_i + \frac{1}{2}\bar{K}_2, t_i + \frac{1}{2}h\right)$$

$$\bar{K}_4 = h\bar{F}(\bar{X}_i + \bar{K}_3, t_{i+1})$$

In Wilson and Newmark method the temporal difference equations are combined with the current equations of motion and the resulting equations are solved to find the current displacement. They belong to the category of implicit integration method.

### WILSON METHOD

- The Wilson method is also known as Wilson  $\theta$  method. If  $\theta=1.0$ , this method reduces to the linear acceleration scheme. It can be described by the following steps:

From the known initial conditions  $\bar{x}_0$  and  $\dot{\bar{x}}_0$ , find  $\ddot{\bar{x}}_0$  using equation

$$\ddot{\bar{x}}_0 = [m]^{-1} (\bar{F}_0 - [c]\dot{\bar{x}}_0 - [k]\bar{x}_0)$$

Select a suitable time step  $\Delta t$  and a suitable value of  $\theta$  ( $\theta$  is usually taken as 1.4).

- In Wilson method it is assumed that the acceleration of the system varies linearly between two instants of time
- Compute the effective load vector  $\bar{F}_{i+\theta}$  starting with  $i=0$ :
- 

$$\begin{aligned} \bar{F}_{i+\theta} = \bar{F}_i + \theta(\bar{F}_{i+1} - \bar{F}_i) + [m] \left( \frac{6}{\theta^2(\Delta t)^2} \bar{x}_i + \frac{6}{\theta\Delta t} \dot{\bar{x}}_i + 2\ddot{\bar{x}}_i \right) \\ + [c] \left( \frac{3}{\theta\Delta t} \bar{x}_i + 2\dot{\bar{x}}_i + \frac{\theta\Delta t}{2} \ddot{\bar{x}}_i \right) \end{aligned}$$

Find the displacement vector at time  $t_{i+\theta}$

$$\bar{x}_{i+\theta} = \left[ \frac{6}{\theta^2(\Delta t)^2} [m] + \frac{3}{\theta\Delta t} [c] + [k] \right]^{-1} \bar{F}_{i+\theta}$$

- Calculate the acceleration, velocity and displacement vectors at time  $t_{i+1}$ :

$$\ddot{\bar{x}}_{i+1} = \frac{6}{\theta^3(\Delta t)^2} (\bar{x}_{i+\theta} + \bar{x}_i) - \frac{6}{\theta^2\Delta t} \dot{\bar{x}}_i + \left( 1 - \frac{3}{\theta} \right) \ddot{\bar{x}}_i$$

$$\dot{\bar{x}}_{i+1} = \dot{\bar{x}}_i + \frac{\Delta t}{2} (\ddot{\bar{x}}_{i+1} + \ddot{\bar{x}}_i)$$

$$\bar{x}_{i+1} = \bar{x}_i + \Delta t \dot{\bar{x}}_i + \frac{(\Delta t)^2}{6} (\ddot{\bar{x}}_{i+1} + 2\ddot{\bar{x}}_i)$$



## NEWMARK METHOD

- This method is also based on the assumption that the acceleration varies linearly between two instant of time.
- The resulting expression for the velocity and displacement respectively can be written for multi degree of freedom as,

$$\dot{\bar{x}}_{i+1} = \dot{\bar{x}}_i + \left[ (1 - \beta) \ddot{\bar{x}}_i + \beta \ddot{\bar{x}}_{i+1} \right] \Delta t$$

$$\bar{x}_{i+1} = \bar{x}_i + \Delta t \dot{\bar{x}}_i + \left[ \left( \frac{1}{2} - \alpha \right) \ddot{\bar{x}}_i + \alpha \ddot{\bar{x}}_{i+1} \right] (\Delta t)^2$$

here  $\alpha$  and  $\beta$  indicates how much the acceleration at the end of the interval enters into the velocity and displacement equations at the end of the interval  $\Delta t$ .

- The equilibrium equation of motion of a viscously damped multi degree of freedom system at  $t = t_{i+1}$  can be written as:

$$[m] \ddot{\bar{x}}_{i+1} + [c] \dot{\bar{x}}_{i+1} + [k] \bar{x}_{i+1} = \bar{F}_{i+1}$$

- Using above equations we can obtain a relation to find  $\bar{x}_{i+1}$

$$\bar{x}_{i+1} = \left[ \frac{1}{\alpha (\Delta t)^2} [m] + \frac{\beta}{\alpha \Delta t} [c] + [k] \right]^{-1}$$

$$\times \left\{ \bar{F}_{i+1} + [m] \left( \frac{1}{\alpha (\Delta t)^2} \bar{x}_i + \frac{1}{\alpha \Delta t} \dot{\bar{x}}_i + \left( \frac{1}{2\alpha} - 1 \right) \ddot{\bar{x}}_i \right) \right.$$

$$\left. + [c] \left( \frac{\beta}{\alpha \Delta t} \bar{x}_i + \left( \frac{\beta}{\alpha} - 1 \right) \dot{\bar{x}}_i + \left( \frac{\beta}{\alpha} - 2 \right) \frac{\Delta t}{2} \ddot{\bar{x}}_i \right) \right\}$$

After obtaining the time response one may be interested to know the frequency content of the time response by performing the Fourier transform using the following equation.

$$x(f) = \int_{-\infty}^{\infty} x(t)e^{-2i\pi ft} dt$$

Assumption that the function is integrable can be checked by using the following

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

Finite Fourier Transform

$$x(f, T_c) = \int_0^{T_c} x(t)e^{-2i\pi ft} dt$$

Power Spectrum/ Autospectrum/ Autospectral density function or Power spectral density function can be obtained by using the following equation.

$$S_{xx}(f) = \int_{-\infty}^{+\infty} R_{xx}(\tau)e^{-2i\pi f\tau} d\tau$$

where

$$R_{xx}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T x(t)x(t + \tau) dt$$

Also it can be obtained by using the following equation

$$S_{xx}(f) = \lim_{T \rightarrow \infty} \frac{1}{T} \left| \int_0^T x(t)e^{-2i\pi ft} dt \right|^2 ..$$

### Exercise Problem

1. Write a Matlab code using Newton Raphsons method to solve the following characteristics equations. Check the results using method of bisections.

(a)  $1 + \cos \beta l \cdot \cosh \beta l = 0$

(b)  $1 - \cos \beta l \cdot \cosh \beta l = 0$

2. Compute the integrals given in equation (5.1.10). You may use the int function of the Matlab. Use dummy variables while performing double and triple integration. Also write a Matlab code

using 12 point Gauss-Quadrature and compare the results. Find the coefficients of the temporal equation of motion.

3. Solve the frequency response equation obtained from the Duffing equation which is quadratic in terms of the detuning parameter and 6<sup>th</sup> order in terms of amplitude of the response.

4. Use Runge-Kutte method (may use ode45 function of Matlab) to obtain the time response of the following equations.

$$(i) 3\ddot{x} + 30x - 0.1x^2 + 0.05x^3 = 0$$

$$(ii) \ddot{x} + 20x - 0.5x^2 + 0.05x^3 = 10 \sin 5t$$

$$(iii) \ddot{x} + 50x - 0.5x^2 + 0.3x^3 = 0.1 \sin 2t$$

$$(iv) \ddot{x} + 50x - (0.1 \sin 2t)x = 0$$

$$(v) \ddot{x} + 50x - 0.25x^2 - (0.1 \sin 2t)x = 0$$

$$(vi) \ddot{x} + 50x + 0.25x^3 + (0.1 \sin 2t)x = 0$$

#### References :

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## Module 5: Lecture 2

### Methods of model reduction and continuation techniques

In this lecture the modal reduction methods used for systems with large degrees of freedom will be discussed. Also, the continuation algorithms used for getting the response of the system will be briefly described.

The equation of a multi-degree of freedom system can be given by

$$M\ddot{x} + C\dot{x} + Kx + f_p(x, \dot{x}, t) + f(x, \dot{x}) = F_{ext}(t) \quad (5.2.1)$$

Here  $M$  is the mass matrix,  $C$  is the damping matrix,  $K$  is the stiffness matrix,  $f_p$  is the forcing term which is a function of displacement, velocity and time,  $f$  is a forcing term which is function of displacement and velocity, and  $F_{ext}$  is the external forcing vector which is a function of time only. This equation may be considered as an equation obtained from the finite element analysis also. Hence the size of these matrices may be very large depending on the number of elements, element type, the degrees of freedom per node and boundary conditions used in the analysis. So it will require very high computational time and memory to solve the full system of equations. Out of these total degrees of freedom as only few degrees of freedom are required for actual analysis purpose, hence one may perform modal reduction to reduce the matrix sizes by performing the following operation. Considering  $M$ ,  $C$  and  $K$  to be  $n \times n$  matrices and letting  $U$

to be the  $n \times n$  modal matrix obtained from the eigenvectors of the  $M^{-1}K$ , taking  $x = U_R z$  and premultiplying  $U_L^T$  in equation (5.2.1) one obtains the following equation.

$$U_L^T M U_R \ddot{z} + U_L^T C U_R \dot{z} + U_L^T K U_R z + U_L^T f_p(z, \dot{z}, t) + U_L^T f(z, \dot{z}) = U_L^T F(t) \quad (5.2.2)$$

Here  $U_L$  and  $U_R$  are the left and right eigenvectors corresponding to  $U$ . Here one may consider only the first  $m$  modes instead of considering all  $n$  modes of the system.

Also one may use dynamic substructuring method in which the mass, stiffness and damping matrices are partitioned into master and slave subsystems. The degrees of freedom where the nonlinearity is present or where the external load is applied are considered as master system and rests are considered as slave system. Taking  $x_s$  and  $x_m$  as the displacement in slave and master degrees of freedom, Eq.(5.2.1) can be written as

$$\begin{aligned} & \begin{bmatrix} M_{ss} & M_{sm} \\ M_{ms} & M_{mm} \end{bmatrix} \begin{Bmatrix} \ddot{x}_s \\ \ddot{x}_m \end{Bmatrix} + \begin{bmatrix} C_{ss} & C_{sm} \\ C_{ms} & C_{mm} \end{bmatrix} \begin{Bmatrix} \dot{x}_s \\ \dot{x}_m \end{Bmatrix} + \begin{bmatrix} K_{ss} & K_{sm} \\ K_{ms} & K_{mm} \end{bmatrix} \begin{Bmatrix} x_s \\ x_m \end{Bmatrix} + \\ & \begin{Bmatrix} 0 \\ f_p(x_m, \dot{x}_m, t) \end{Bmatrix} + \begin{Bmatrix} 0 \\ f(x_m, \dot{x}_m) \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_m \end{Bmatrix} \end{aligned} \quad (5.2.3)$$

Here,

$$M = \begin{bmatrix} M_{ss} & M_{sm} \\ M_{ms} & M_{mm} \end{bmatrix}; C = \begin{bmatrix} C_{ss} & C_{sm} \\ C_{ms} & C_{mm} \end{bmatrix}; K = \begin{bmatrix} K_{ss} & K_{sm} \\ K_{ms} & K_{mm} \end{bmatrix}. \quad (5.2.4)$$

Now one may apply harmonic balance method where the  $k^{\text{th}}$  element of the response vector, external force vector and nonlinear functions are expressed as a truncated Fourier series of  $n$  terms as given below.

$$x_k(t) = \tilde{x}_{k0} + \sum_{l=1}^n \left\{ \tilde{x}_{kl}^c \cos(l\omega t) + \tilde{x}_{kl}^s \sin(l\omega t) \right\} \quad (5.2.5)$$

$$F_k(t) = \tilde{F}_{k0} + \sum_{l=1}^n \left\{ \tilde{F}_{kl}^c \cos(l\omega t) + \tilde{F}_{kl}^s \sin(l\omega t) \right\} \quad (5.2.6)$$

$$f_k(x, \dot{x}) = \tilde{f}_{k0} + \sum_{l=1}^n \left\{ \tilde{f}_{kl}^c \cos(l\omega t) + \tilde{f}_{kl}^s \sin(l\omega t) \right\} \quad (5.2.7)$$

$$f_{pk}(x, \dot{x}) = \tilde{f}_{pk0} + \sum_{l=1}^n \left\{ \tilde{f}_{pkl}^c \cos(l\omega t) + \tilde{f}_{pkl}^s \sin(l\omega t) \right\} \quad (5.2.8)$$

where  $k = 1, 2, \dots, N$  with  $N$  being the number of degrees of freedom.

Substituting Eqs. (5.2.5)-(5.2.8) in Eq. (5.2.1) one can obtain the following equation.

$$\begin{bmatrix} E_{11} & E_{12} & \cdots & E_{1N} \\ E_{21} & E_{22} & \cdots & E_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ E_{N1} & E_{N2} & \cdots & E_{NN} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_N \end{bmatrix} + \begin{bmatrix} \tilde{f}_{P1} \\ \tilde{f}_{P2} \\ \vdots \\ \tilde{f}_{PN} \end{bmatrix} + \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \vdots \\ \tilde{f}_N \end{bmatrix} - \begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ \vdots \\ \tilde{F}_N \end{bmatrix} = R(\gamma) \quad (5.2.9)$$

Here  $R(\gamma)$  is the residual error due to truncation in the Fourier series.  $\tilde{x}_i, \tilde{f}_{pi}, \tilde{f}_i$ , and  $\tilde{F}_i$  represents the  $i^{\text{th}}$  vector of constant term, cosine term, sine term in the sequence of particular degree of freedom.

$$E_{ij} = \begin{bmatrix} k_{ij} & 0 & 0 & \cdots & 0 & 0 \\ 0 & k_{ij} - \omega^2 m_{ij} & \omega c_{ij} & \cdots & 0 & 0 \\ 0 & -\omega c_{ij} & k_{ij} - \omega^2 m_{ij} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & k_{ij} - n^2 \omega^2 m_{ij} & n \omega c_{ij} \\ 0 & 0 & 0 & \cdots & -n \omega c_{ij} & k_{ij} - n^2 \omega^2 m_{ij} \end{bmatrix} \quad (5.2.10)$$

Now one may minimize the residual as

$$\sum_{n=1}^m \left[ \int_0^{2\pi/\omega} R(\gamma) \begin{pmatrix} 1 \\ \cos(n\omega t) \\ \sin(n\omega t) \end{pmatrix} dt \right] = 0 \quad (5.2.11)$$

to obtain

$$Y\tilde{x}_a + \tilde{f}_p + \tilde{f} - \tilde{F} = 0 \quad (5.2.12)$$

where

$$Y = \begin{bmatrix} E_{11} & E_{12} & \cdots & E_{1N} \\ E_{21} & E_{22} & \cdots & E_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ E_{N1} & E_{N2} & \cdots & E_{NN} \end{bmatrix} \quad (5.2.13)$$

Now one can solve the algebraic equation (5.2.12) by using Newton-Raphson method where the Jacobian can be given by

$$J = \frac{\partial(Y\tilde{x}_a + \tilde{f}_p + \tilde{f} - \tilde{F})}{\partial\tilde{x}_a} \quad (5.2.14)$$

To apply this method to the equation obtained by dynamic substructuring technique one may obtain the following nonlinear algebraic equation.

$$\begin{bmatrix} Y_{ss} & Y_{sm} \\ Y_{ms} & Y_{mm} \end{bmatrix} \begin{Bmatrix} \tilde{x}_s \\ \tilde{x}_m \end{Bmatrix} + \begin{Bmatrix} 0 \\ \tilde{f}_p \end{Bmatrix} + \begin{Bmatrix} 0 \\ \tilde{f} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \tilde{F} \end{Bmatrix} \quad (5.2.15)$$

Here  $\tilde{x}_s$  represents the coefficients from harmonic balance method (HBM) for the slave degrees of freedom and  $\tilde{x}_m$  represents the coefficients for the master degrees of freedom. Here  $\tilde{f}$  and  $\tilde{F}$  are the coefficients from HBM for nonlinear and external force terms. From the first row of Eq. (5.2.15) one obtains

$$\tilde{x}_s = -(Y_{ss}^{-1}Y_{sm})\tilde{x}_m \quad (5.2.16)$$

Now using this expression in the 2<sup>nd</sup> row of Eq. (5.2.15) the following equation is obtained.

$$(Y_{mm} - Y_{ms}Y_{ss}^{-1}Y_{sm})\tilde{x}_m + \tilde{f}_p + \tilde{f} = \tilde{F} \quad (5.2.17)$$

One may note that the size of this equation (5.2.17) is smaller than that of the original equation and hence required less memory and computational time.

While plotting the frequency response curves of the nonlinear system often it is required to adopt a continuation scheme as multiple branches are observed in the frequency response curves. Few continuation schemes are given below.

### Sequential Continuation

The simplest continuation method is sequential scheme and also known as natural parameter continuation. Consider a variation of solutions of

$$F(x; \alpha) = 0 \quad (5.2.18)$$

where  $\alpha$  is used as the continuation parameter. The interval of  $\alpha$  is divided into closely spaced  $n$  intervals defined by the grid points  $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n$ . Then the solution  $x_j$  at  $\alpha_j$  is used as the predicted value or initial guess for the solution  $X_{j+1}$  at  $\alpha_{j+1}$ . The predicted value corrected through a Newton-Raphson iteration scheme. Thus

$$x_{j+1}^{k+1} = x_{j+1}^k + r\Delta x^k \quad (5.2.19)$$

Here the subscript  $k$  is the iteration number and  $x_{j+1}^1 = x_j$  and  $r$  is called the relaxation parameter, such that  $0 < r \leq 1$  and  $\Delta x^k$  is the solution of the  $n$  linear algebraic equations

$$F_x(x_{j+1}^k, \alpha_{j+1})\Delta x^k = -F(x_{j+1}^k, \alpha_{j+1}) \quad (5.2.20)$$

obtained using Newton-Raphson method. The relaxation parameter  $r$  is chosen such that

$$\|F(x_{j+1}^{k+1}, \alpha_{j+1})\| < \|F(x_{j+1}^k, \alpha_{j+1})\| \quad (5.2.21)$$

- If the above relation is satisfied for  $r=L$ , then  $r$  is set equal to 1 otherwise  $r$  is halved until the above equation is satisfied.
- If the grid points are sufficiently close, few iterations are sufficient for obtaining an accurate solution.
- Sequential continuation will fail at such points where two or more branches meet because the Jacobian is singular there.

### Dauidenko-Newton-Raphson continuation (Nayfeh and Balachandran 1995)

In this method a method developed by Dauidenko is used as a predictor and then Newton-Raphson method is used as a corrector. As the predictor is based on solving a system of ordinary-differential equations, hence it is also called an *ordinary-differential equation predictor*. Considering  $\alpha$  as the continuation parameter and differentiating (5.2.18) with respect to  $\alpha$  one obtains

$$F_x(x, \alpha) \frac{dx}{d\alpha} = -F_\alpha(x, \alpha) \quad (5.2.22)$$

This constitutes a system of linear algebraic equations for the unknowns  $dx/d\alpha$

Hence, if the Jacobian  $F_x(x, \alpha)$  is regular in the interval  $[\alpha_0, \alpha_n]$ , then

$$\frac{dx}{d\alpha} = -F_x^{-1}(x, \alpha) F_\alpha(x, \alpha) \quad (5.2.23)$$

Then, the given solution  $x(\alpha_0) = x_0$ , one can solve Eq. (5.2.23) by using any ordinary differential solver e.g., Runge-Kutta method. The predicted values obtained from the integration are likely to deviate from the true solutions due to truncated error. Hence, the predicted values are used as initial guesses for a Newton-Raphson method to obtain corrected values.

### Arc length Continuation

- In this method, the arc length  $s$  along a branch of solutions is used as the continuation parameter.
- $x$  and  $\alpha$  are considered to be function of  $s$ ; that is,  $x=x(s)$  and  $\alpha = \alpha(s)$  such that
 
$$F[x(s), \alpha(s)] = 0 \quad (5.2.24)$$

Differentiating Eq. (5.2.24) with respect to  $s$  one obtains

$$F_x(x, \alpha) \frac{dx}{ds} + F_\alpha(x, \alpha) \frac{d\alpha}{ds} = 0 \quad (5.2.25)$$

$$\text{Or, } F_x(x, \alpha)x' + F_\alpha(x, \alpha)\alpha' = 0 \quad (5.2.26)$$



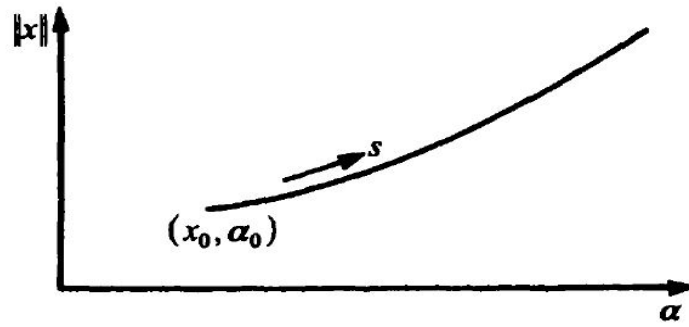


Fig. 5.2.1: Illustration for the arc length continuation scheme

Equation (5.2.25) may be rewritten as  $[F_x | F_\alpha]t = 0$  where  $t = \left\{ \frac{dx}{ds} \quad \frac{d\alpha}{ds} \right\}' = \{x' \quad \alpha'\}'$ . Here the  $(n+1)$  vector  $t$  is the tangent vector at  $(x, \alpha)$  on the path. The system (5.2.25) consists of  $n$  linear algebraic equations in the  $(n+1)$  unknowns. To specify this unknowns uniquely, we supplement (5.2.25) with a homogeneous equation specified by Euclidean arclength normalization

$$x'^T x' + \alpha'^2 = x_1'^2 + x_2'^2 + \dots + x_n'^2 + \alpha'^2 = 1 \quad (5.2.27)$$

The initial conditions for (5.2.25) and (5.2.26) are given by

$$x = x_0 \text{ and } \alpha = \alpha_0 \quad \text{at } s=0 \quad (5.2.28)$$

If the Jacobian  $F_x$  is nonsingular and  $F_\alpha$  is a zero vector, (5.2.24) and (5.2.27) gives rise to

$$\begin{bmatrix} x'^T & \alpha' \end{bmatrix} = \pm [0 \ 0 \ \dots \ 0 \ 1]$$

If the Jacobian is non singular and  $F_\alpha$  is a nonzero vector, one can solve (5.2.24) and (5.2.27) to determine the tangent vector  $t$ . Then it is used to predict the values of  $x$  and  $\alpha$  at  $s + \Delta s$  by taking an Euler step  $x = x_0 + x' \Delta s$  and  $\alpha = \alpha_0 + \alpha' \Delta s$ .

### References

1. Praveen Krishna I.R., Experimental and Numerical studies of Mechanical Systems with Localized Nonlinearities, PhD thesis, Indian Institute of Technology Madras, 2011
2. A. H. Nayfeh and B. Balachandran: Applied nonlinear Dynamics, Wiley, 1995
3. I. R. Praveen Krishna and C. Padmanabhan: Improved reduced order solution techniques for nonlinear systems with localized nonlinearities. Nonlinear Dynamics, 63(4), 561-586, 2011.
4. R. Seydel: From Equilibrium to Chaos, Elsevier, 1988
5. T. S. Parker and L.O. Chua: Practical Numerical Algorithms for Chaotic System

### Exercise problems

1. Use the studied method to solve the following equation which represents oscillations of two mass system with linear and spring damper system

$$\begin{bmatrix} 12 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 0.2 & -0.2 \\ -0.2 & 0.2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 18 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0.01(x_1 - x_2)^2 \\ -0.01(x_1 - x_2)^2 \end{Bmatrix} = \begin{Bmatrix} p \cos \omega t \\ 0 \end{Bmatrix}$$

Use the continuation methods to plot the frequency response curves.

### Module 5 Lecture 3

#### Poincare section, basin of attraction and Liapunov exponent

In this lecture determination of Poincare section, basin of attraction and Liapunov exponent will be discussed.

#### Poincare' Section

Let us consider a periodic response  $y = 5 \sin 2t$ . The time response and phase portrait for this response is shown in Fig. 5.3.1. If one plot the phase portrait by sampling at time interval  $T = 2\pi/2 = \pi$ , it will yield a single point as shown in Fig. 5.3.1(c). It may be noted that the actual phase portrait is shown in Fig. 5.3.1(b). The plot shown in Fig. 5.3.1(c) is called the Poincare' section of the periodic response shown in Fig. 5.3.1(a). So, in this way one can determine the Poincare' section by sampling the time response with minimum time period. It is a method used to reduce the dimension of the system by one.

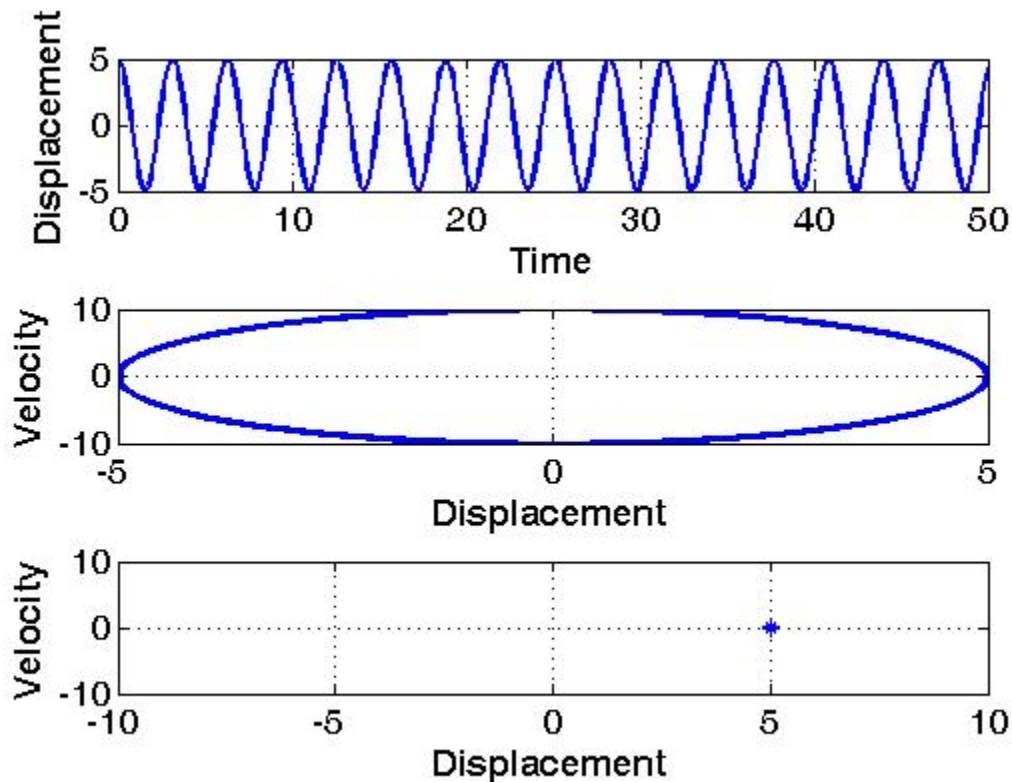


Fig. 5.3.1: (a) time response, (b) phase portrait (c) Poincare section.

The Matlab code for plotting figure 5.3.1 is given below.

```

clc
clear all
a0=5;
t=0:0.1:50;
omega=2;
beta=0;
u=a0*cos(omega*t+beta);
ut=-a0*omega*sin(omega*t+beta);

subplot(3,1,1)
plot(t,u)
% title('Time Response')
set(findobj(gca,'Type','line'),'Color','b','LineWidth',2);
set(gca,'FontSize',14)
xlabel('Time','fontSize',14,'fontWeight','b');
ylabel('Displacement','fontSize',14,'fontWeight','b');
grid on

subplot(3,1,2)
plot(u,ut)
% title('Phase portrait')
set(findobj(gca,'Type','line'),'Color','b','LineWidth',2);
set(gca,'FontSize',14)

```

```

xlabel('Displacement','fontsize',14,'fontweight','b');
ylabel('Velocity','fontsize',14,'fontweight','b');
%ylabel('u\dot','fontsize',14,'fontweight','b');
grid on

%Poincare section

w=2*pi/omega; % Sampled with minimum time Period
tp=0:w:50;
up=a0*cos(omega*tp+beta);
utp=-a0*omega*sin(omega*tp+beta);
subplot(3,1,3)
plot(up,utp,'*')
axis([-10 10 -10 10])
% title('Poincare section')
%set(findobj(gca,'Type','point'),'Color','b','LineWidth',2);
set(gca,'FontSize',14)
xlabel('Displacement','fontsize',14,'fontweight','b');
ylabel('Velocity','fontsize',14,'fontweight','b');
%ylabel('u\dot','fontsize',14,'fontweight','b');
grid on

```

Let us take another example. Here, let  $y = 5(\cos 2t + \cos 2\sqrt{2}t)$ . Here the system has two time period, viz.,  $T_1 = 2\pi/2 = \pi$  and  $T_2 = 2\pi/2\sqrt{2} = \pi/\sqrt{2}$ . By taking the sampling time as  $T_2$  (which is the minimum time period) one may plot the phase portrait as shown in Fig. 5.3.2 (c). Clearly one may observe a close loop. The Matlab code for this purpose is given below.

```

% Quasi-Periodic response (Poincare section)
clc
clear all
a0=5;
t=0:0.01:50;
omega1=2;
omega2=2*sqrt(2);
%beta=-3.15;
beta=0;
u=a0*(cos(omega1*t)+cos(omega2*t));
ut=-a0*(omega1*sin(omega1*t)+omega2*sin(omega2*t));

subplot(3,1,1)
plot(t,u)
% title('SYSTEM WITH LINEAR DAMPING')
set(findobj(gca,'Type','line'),'Color','b','LineWidth',2);
set(gca,'FontSize',14)
xlabel('Time','fontsize',14,'fontweight','b');
ylabel('Displacement','fontsize',14,'fontweight','b');
grid on

subplot(3,1,2)
plot(u,ut)

```

```

% title('SYSTEM WITH LINEAR DAMPING')
set(findobj(gca, 'Type', 'line'), 'Color', 'b', 'LineWidth', 2);
set(gca, 'FontSize', 14)
xlabel('Displacement', 'fontsize', 14, 'fontweight', 'b');
ylabel('Velocity', 'fontsize', 14, 'fontweight', 'b');
%ylabel('u\dot', 'fontsize', 14, 'fontweight', 'b');
grid on

%Poincare section
%figure(2)
w=2*pi/omega2;
tp=0:w:500;
up=a0*(cos(omega1*tp)+cos(omega2*tp));
utp=-a0*(omega1*sin(omega1*tp)+omega2*sin(omega2*tp));
subplot(3,1,3)
plot(up,utp, '.')
%axis([-10 10 -10 10])
% title('Poincare section')
%set(findobj(gca, 'Type', 'point'), 'Color', 'b', 'LineWidth', 2);
set(gca, 'FontSize', 14)
xlabel('Displacement', 'fontsize', 14, 'fontweight', 'b');
ylabel('Velocity', 'fontsize', 14, 'fontweight', 'b');
%ylabel('u\dot', 'fontsize', 14, 'fontweight', 'b');
grid on

```

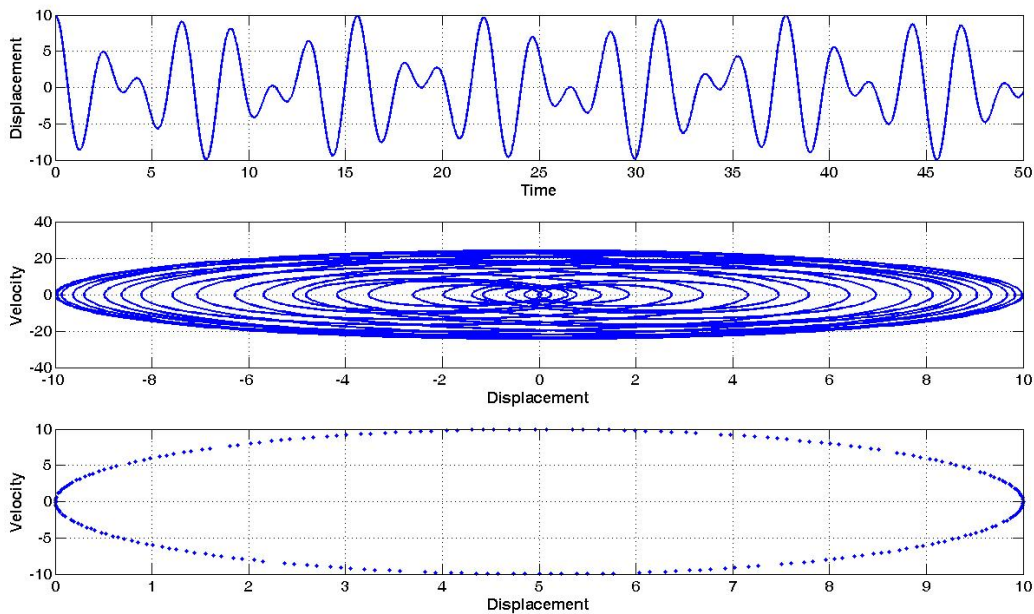
Hence, for quasi-periodic response the Poincare' section is a close curve in phase portrait. For chaotic response the Poincare' section fills of the whole space in the phase portrait. In Fig.

5.3.3, taking a displacement of the form  $y = \sum_{n=0}^6 5 \cos(2^n t)$ , the time response, phase portrait and

Poincare' section have been plotted. Here, the time is sampled in the interval of  $T = 2\pi / 64 = \pi / 32$ . Unlike the case of periodic and quasi-periodic response, here the Poincare' section fills up the phase portrait.

Quasi-Periodic

response



Quasi-Periodic Response

Fig 5.3.2 (a) time response (b) Phase Portrait (c) Poincaré' section

```

% chaotic response (Poincare section)
clc
clear all
a0=5;

omega=1;
u=0;
ut=0;
up=0;
utp=0;

t=0:0.01:50;
ii=1;
for ip=1:1:7;
u=u+a0*cos(ii*omega*t);
ut=ut+(-a0*(ii*omega*sin(ii*omega*t)));
ii=2^ip
end
subplot(3,1,1)
plot(t,u)
% title('SYSTEM WITH LINEAR DAMPING')
set(findobj(gca,'Type','line'),'Color','b','LineWidth',2);
set(gca,'FontSize',14)
xlabel('Time','fontsize',14,'fontweight','b');
ylabel('Displacement','fontsize',14,'fontweight','b');
grid on

```

```

subplot(3,1,2)
plot(u,ut)
% title('SYSTEM WITH LINEAR DAMPING')
set(findobj(gca, 'Type', 'line'), 'Color', 'b', 'LineWidth', 2);
set(gca, 'FontSize', 14)
xlabel('Displacement', 'fontsize', 14, 'fontweight', 'b');
ylabel('Velocity', 'fontsize', 14, 'fontweight', 'b');
%ylabel('u\dot', 'fontsize', 14, 'fontweight', 'b');
grid on

%Poincare section
%figure(2)
w=2*pi/(2^9*omega);
tp=0:w:500;
ir=1;
for ii=1:1:7;
up=up+a0*cos(ir*omega*tp);
utp=utp+(-a0*(ir*omega*sin(ir*omega*tp)));
ir=2^ii
end

subplot(3,1,3)
plot(up,utp, '.')
%axis([-10 10 -10 10])
% title('Poincare section')
%set(findobj(gca, 'Type', 'point'), 'Color', 'b', 'LineWidth', 2);
set(gca, 'FontSize', 14)
xlabel('Displacement', 'fontsize', 14, 'fontweight', 'b');
ylabel('Velocity', 'fontsize', 14, 'fontweight', 'b');
%ylabel('u\dot', 'fontsize', 14, 'fontweight', 'b');
grid on

```

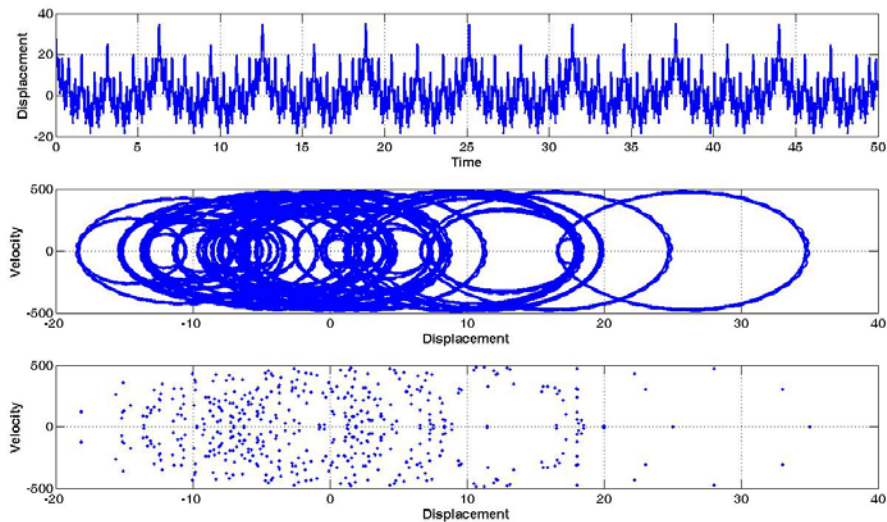


Fig. 5.3.3: (a) time response (b) phase portrait (c) Poincare section for a chaotic response.

In the above examples, as the minimum period of the response is known, one can sample the response easily to obtain the Poincare' section. But in most of the cases, while we study the response of a system, the periods or response frequencies are not known *a priori*. In that case one may take the first Fourier transform of the response to obtain the minimum period which corresponds to the maximum frequency of the system.

### Basin of Attraction

Consider a typical frequency response plot shown in Fig. 5.3.4 which is obtained in Duffing type oscillator with soft nonlinearity. The black and red curves are plotted for two values of the coefficients of cubic nonlinear terms.

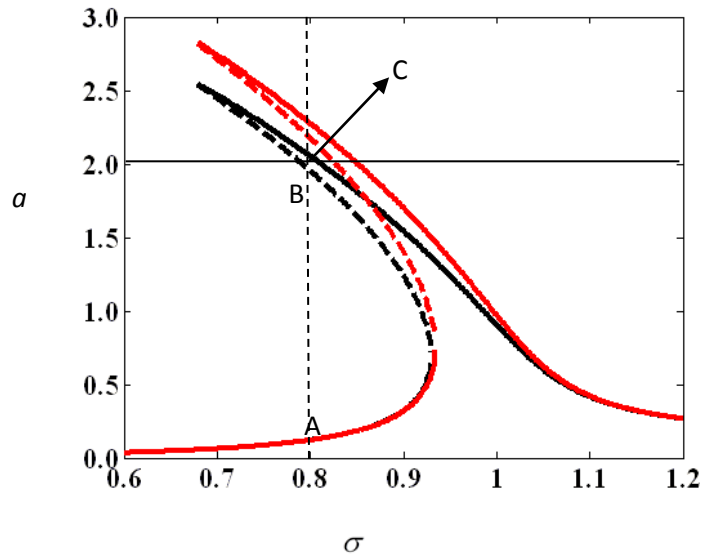


Fig. 5.3.4: Typical Phase portrait showing existence of multiple solutions



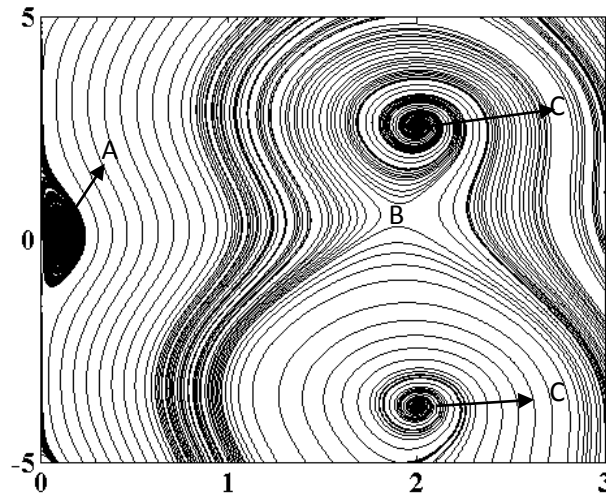


Fig. 5.3.5: Basin of attraction

Corresponding to  $\sigma = 0.8$ , one may observe that there exists two stable and one unstable response of the system. The stable responses are marked by point A and C and the unstable response is shown by point B in Fig. 5.3. 4. In practical application, to know which response is actually occurring, one should plot the basin of attraction as shown in Fig. 5.3.5. It is clearly shown that for some initial condition one may observe point A and for some other initial conditions one will obtain point C. As point B is a unstable point, in the basin of attraction this point is shown as a saddle point. This plot can be obtained by taking many initial points and plotting the phase portraits. This brute-force method is very time consuming and one may use several other technique like point-to-point mapping or cell-to-cell-mapping techniques to obtain the basin of attraction. Following algorithm may be used to plot basin of attraction.

1. Divide the domain of phase portrait into different cells by properly choosing the interval of the required state vectors.
2. Number these cells, say 1, 2, ..., n.
3. Determine the center of the cells.
4. Taking the center of the 1<sup>st</sup> cell as as the initial point, solve the governing equation to obtain the trajectory. One may use numerical technique like Runge-Kutta method (ode45 function of Matlab) to find the trajectory.
5. Note the cell through which the trajectory is passing. Note the point where the trajectory is terminating. This corresponds to the steady state response or equilibrium point.

6. Considering the next cell, repeat step 4.
7. If the trajectory passes through the same cells for a considerable amount of time, it may be assumed that with this initial condition, one will obtain the first equilibrium point. Otherwise it may lead to some other equilibrium point.
8. Repeat step 7, till all the cells have been marked with their corresponding equilibrium point.

### Lyapunov exponent

Lyapunov' exponents or characteristic exponents associated with a trajectory are a measure of the average rates of expansion and contraction of the trajectories surrounding it. **Any system containing at least one positive Lyapunov exponent is defined to be chaotic. The magnitude of the exponent reflecting the time scale on which system dynamics become unpredictable.**

### Determination of Lyapunov exponent

Consider the dynamical system

$$\dot{x} = F(x; M)$$

$$x(t) = x_0 + y(t)$$

$$\dot{y} = F(x_0 + y; M)$$

$$\dot{y} = F(x_0 + y; M) + D_x F(x_0; M) y + O(\|y\|^2)$$

$$\dot{y} = D_x F(x_0; M_0) y \equiv Ay$$

where

$$A = \begin{bmatrix} \frac{dF_1}{dx_1} & \frac{dF_1}{dx_2} & \dots & \frac{dF_1}{dx_n} \\ \frac{dF_2}{dx_1} & \frac{dF_2}{dx_2} & \dots & \frac{dF_2}{dx_n} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{dF_n}{dx_1} & \frac{dF_n}{dx_2} & \dots & \frac{dF_n}{dx_n} \end{bmatrix}$$

$$\dot{y} = Ay(t) \quad (4)$$

Taking an initial deviation  $y(0)$ , its evolution can be expressed as

$$y(t) = \phi(t)y(0)$$

Here  $\phi(t)$  is the fundamental matrix solution of (4)

The Lyapunov function  $\lambda_i$  can be given by

$$\bar{\lambda}_i = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \left( \frac{\|y(t)\|}{\|y(0)\|} \right)$$

Gram-Schmidt Procedure for orthonormalized vector

$$\hat{y}_1 = \left( \frac{y_1(T)}{\|y_1(T)\|} \right)$$

$$\hat{y}_2 = \frac{y_2(T) - [y_2(T) \cdot \hat{y}_1] \hat{y}_1}{\|y_2(T) - [y_2(T) \cdot \hat{y}_1] \hat{y}_1\|}$$

$$\hat{y}_m = \frac{y_m(T) - \sum_{i=1}^{m-1} [y_m(T) \cdot \hat{y}_i] \hat{y}_i}{\|y_m(T) - \sum_{i=1}^{m-1} [y_m(T) \cdot \hat{y}_i] \hat{y}_i\|}$$

$$\bar{\lambda}_i = \frac{1}{rT} \sum_{k=1}^r \ln N_i^k$$

System	Parameter Values	Lyapunov Spectrum
Henon:		
$X_{n+1} = 1 - aX_n^2 + Y_n$	$a = 1.4$	$\lambda_1 = 0.603$
$Y_{n+1} = bX_n$	$b = 0.3$	$\lambda_2 = -2.34$
Rossler-Chaos:		
$\dot{X} = -(Y + Z)$	$a = 0.15$	$\lambda_1 = 0.13$
$\dot{Y} = X + aY$	$b = 0.20$	$\lambda_2 = 0.00$
$\dot{Z} = b + Z(X - c)$	$c = 10.0$	$\lambda_3 = -14.1$
Lorenz:		

$\dot{X} = \sigma(Y - X)$	$\sigma = 16.0$	$\lambda_1 = 2.16$
$\dot{Y} = X(R - Z)$	$R = 45.92$	$\lambda_2 = 0.00$
$\dot{Z} = XY - bZ$	$b = 4.0$	$\lambda_3 = -32.4$

Rosler-hyperchaos:

$\dot{X} = -(Y + Z)$	$a = 0.25$	$\lambda_1 = 0.16$
$\dot{Y} = X + aY + W$	$b = 3.0$	$\lambda_2 = 0.03$
$\dot{Z} = b + XZ$	$c = 0.05$	$\lambda_3 = 0.00$
$\dot{W} = cW - dZ$	$d = 0.5$	$\lambda_4 = -39.0$

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