

STABILITY AND BIFURCATION ANALYSIS

Lect1: Stability analysis of fixed point response

Lect2: Bifurcation analysis of fixed point response

Lect3: Stability analysis of periodic response: Floquet Analysis

Lect4: Limit cycles and Bifurcation analysis of periodic response

Lect5: Analysis of Quasi periodic and Chaotic response

Module 4 Lecture 1**STABILITY ANALYSIS OF FIXED POINT RESPONSE**

In the previous lecture, we have learned about different perturbation methods to obtain the solution of the nonlinear differential equations of motion. Unlike linear system, where only one solution exists, in nonlinear case one may observe multiple solutions. Also, the solution may be a fixed-point response; it may be periodic, quasi-periodic or chaotic in nature. Out of these multiple solutions, some solution may be stable, other may be unstable. A stable solution is one, which remain bounded when the response is slightly perturbed. If the response grows with slight perturbation, the response is unstable. In this module the stability and bifurcation analysis of different types of responses will be discussed. In this lecture the stability of fixed point response will be discussed.

Let us start with a simple linear system by considering a simple spring-mass system. The governing equation in this case is

$$m\ddot{x} + kx = f \quad (4.1.1)$$

If no external force is acting on the system, the system response is periodic with a frequency $\omega_n = \sqrt{k/m}$ and the amplitude depends on the initial displacement x_0 and velocity \dot{x}_0 . Now taking $f = k_p x$, if $k_p < k$ or $k' (= k - k_p) > 0$, the system will have a bounded solution and when $k' (= k - k_p) < 0$, the response will grow. Taking numerical example of $m = 1 \text{ kg}$, $k = 105 \text{ N/m}$, $k_p = 5 \text{ N/m}$.

Figure 4.1.1 shows the time response and phase portrait.

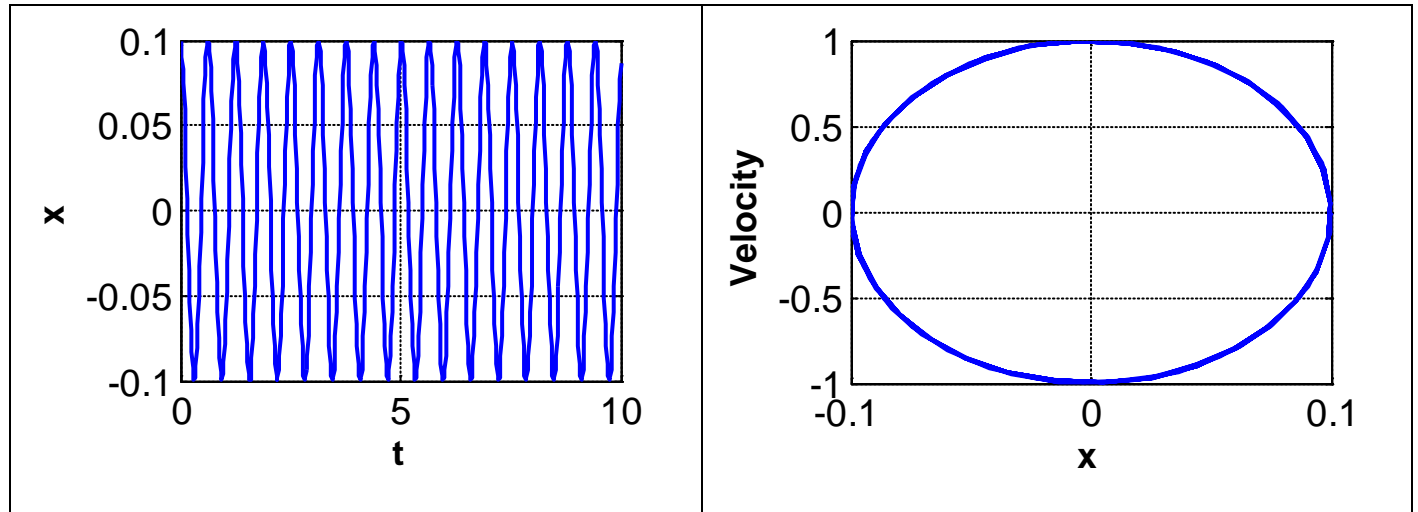


Fig. 4.1.1: Time response and Phase portrait using Eq. (4.1.1) with $m = 1 \text{ kg}, k = 105 \text{ N/m}, k_p = 5 \text{ N/m}$
 $x_0 = 0.1, \dot{x}_0 = 0$

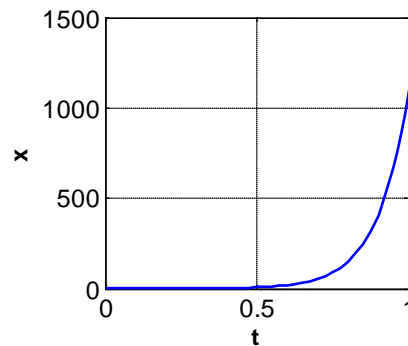


Fig 4.1.2: Time response of the system with $k_p = 205 \text{ N/m}$, other parameters same as in Fig.4. 1.1.

Now taking $m = 1 \text{ kg}, k = 105 \text{ N/m}, k_p = 205 \text{ N/m}$, as $k' < 0$, the response grows with time exponentially as shown in the time response in Fig. 4.1.2. While the system shown in Figure 4.1.1 is said to be stable, that shown in Figure 4.1.2 is unstable. Now by adding damping to the system the periodic response becomes a fixed point response as shown in Figure 4.1.3.

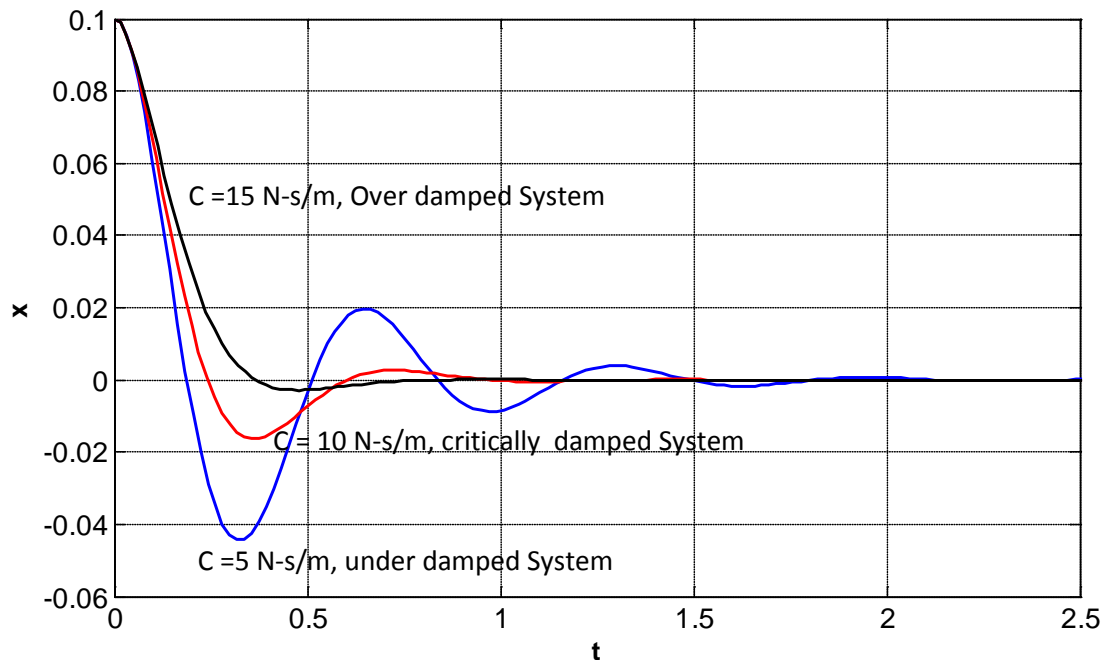


Fig. 4.1.3: Time response curves showing the effect of addition of damping to the system in Eq. (4.1.1) to obtain the fixed-point response

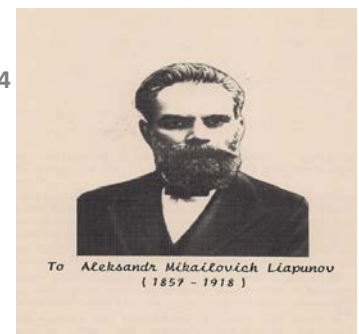
Similar to the response discussed in the linear case, in nonlinear case also, one may check, whether the response is stable or unstable and one should avoid the conditions for which the system becomes unstable. As the system works in a wide range of system parameters, one may be interested to know whether the system is stable or unstable in all the range of working parameters and in that case one should study the global stability of the system. But in case one is interested to know the stability of the system for some specific system parameter, one may perform the local stability analysis. Generally Liapunov direct method is used for studying the global stability of the system and several other linearization techniques are used for studying local stability of the system.

Matlab code for the plot of Figure 4.1.1 and 4.1.2

```
clc
clear all

[T,Y] = ode45(@vdp1000,[0 20],[0.1 0]);
figure(1)

plot(T,Y(:,1),'-')
```



```
figure(2)
plot(Y(:,1),Y(:,2),'-');

function dy = vdp1004(t,y)
dy = zeros(2,1); % a column vector
m=1;
k=105;
kp=5; (%Kp=205 for fig.4.1.2)
dy(1) = y(2);
% dy(2) = 0.5*(1 - y(1)^2)*y(2) - y(1);
dy(2)=-((k-kp)/m)*y(1);
```

http://www.truthscience.info/consistent_lyapunov_methodology

In general let us consider dynamic equation of a system as

$$\dot{x} = f(x, u; M) \tag{4.1.2}$$

where x is the state vector, u is the input vector and M is the control parameter of the system. For constant input parameter, at steady state, i.e., at time $t \rightarrow \infty$, the response of the system is the equilibrium point of the system. Hence to obtain equilibrium point the vector field $f(x, u; M)$ should vanish. So for equilibrium or fixed point response

$$f(x, u; M) = 0 \tag{4.1.3}$$

To study the stability of the equilibrium point few stability criteria are described below.

Liapunov Stability

A stationary solution \hat{x} is said to be asymptotically stable if the response to a small perturbation approached zero as the time approached infinity. An asymptotically stable equilibrium is also called *sink*. Alternatively it can be defined as follows.

An equilibrium point \hat{x} of the system S , is asymptotically stable if and only if for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $\|x(0) - \hat{x}\| < \delta$, then $\|x(t) - \hat{x}\| < \epsilon$ for $t \geq 0$ and $x(t) \rightarrow \hat{x}$ as $t \rightarrow \infty$. This is depicted in Fig. 4.1.4 where starting from an initial point $x(0)$, which is inside the boundary $B(x, \delta)$, as time $t \rightarrow \infty$ the trajectory returns back to a point \hat{x} which is inside the same boundary. Also, the whole trajectory $x(t)$ remain within a finite boundary $B(x, \epsilon)$.

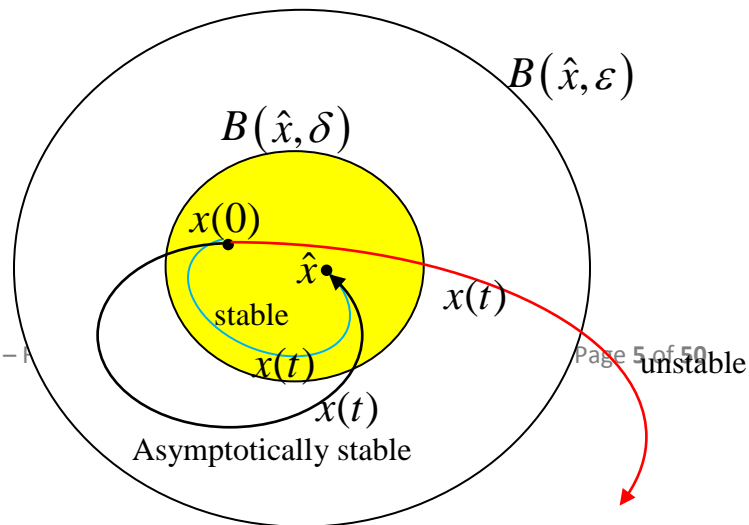


Figure 4.1.4: Stable, unstable and asymptotic stable solution.

A stationary solution \hat{x} is said to be stable if the response to a small perturbation remains small as the time approaches infinity, otherwise the stationary solution is called unstable as in this case the deviation grows with time. An unstable equilibrium is also called a source and is an example of a repeller.

There is a simple sufficient condition proposed by Liapunov which can be used to test an equilibrium point for asymptotic stability. It can be obtained by finding the Jacobian matrix of $f(x, u; M)$. The element of Jacobian matrix can be defined as

$$J(i, j) = \frac{\partial}{\partial x_j} f_k(x) \quad (4.1.4)$$

Liapunov's first method (indirect method)

According to this method the system is asymptotically stable, if the real part of each eigenvalue of the Jacobian matrix is negative.

Hence, if λ_k is the k^{th} eigenvalue of the $m \times m$ Jacobian matrix corresponding to the equilibrium point, then the system is stable if

$$\text{Re}(\lambda_k) < 0, 1 \leq k \leq m \quad (4.1.5)$$

Example 4.1.1:

Find the Jacobian matrix and study the stability of the following two dimensional system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u(5 - x_2) - 2x_1 \end{aligned} \quad (4.1.6)$$

Taking constant input $u(t) = r$ for $t \geq 0$, substituting $\dot{x}_1 = \dot{x}_2 = 0$, the system has a single equilibrium point at $\hat{x} = [2.5r \ 0]$, i.e., $\dot{x}_1 = 2.5r$, $x_2 = 0$. The Jacobian matrix corresponding to this equilibrium point can be found out from the Jacobian matrix

$$J = \begin{bmatrix} 0 & 1 \\ -2 & -r \end{bmatrix} \quad (4.1.7)$$

It may be noted that, in this case the Jacobian matrix does not explicitly depend on the equilibrium point $\hat{x} = [2.5r \ 0]$, but it depends on the input parameter. To find the stability of the system

$$[J - \lambda I] = \begin{bmatrix} -\lambda & 1 \\ -2 & -r - \lambda \end{bmatrix} \quad (4.1.8)$$

$$\lambda^2 + r\lambda + 2 = 0 \quad (4.1.9)$$

$$\lambda_{1,2} = -\frac{r}{2} \pm \frac{\sqrt{r^2 - 8}}{2} \quad (4.1.10)$$

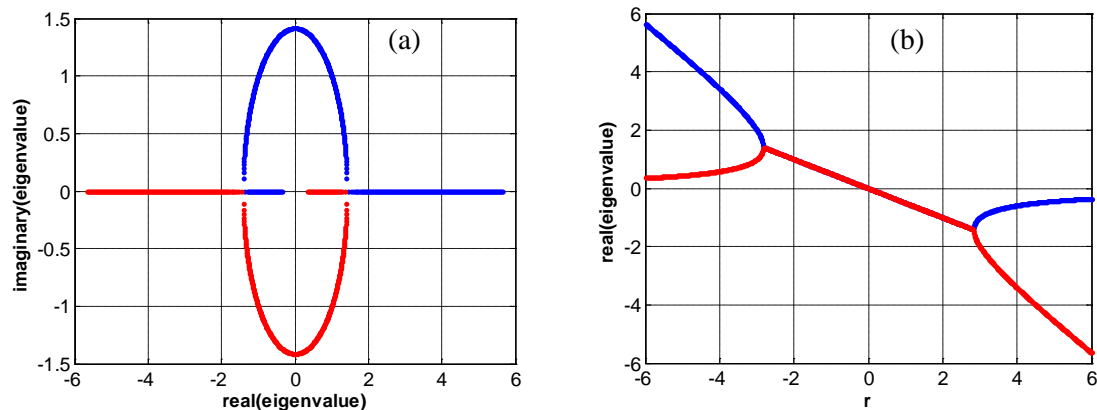


Fig. 4.1.5: (a) variation of real part and imaginary part of eigenvalue. (b) variation of real part of eigenvalue with control parameter r .

Fig. 4.1.5 (a) shows the variation of real part and imaginary part of the eigenvalues with variation in the control parameter r . Fig. 4.1.5 (b) clearly shows that for $r > 0$ the real part of both the eigenvalues are negative.

Liapunov's second method (direct method):

Each asymptotically stable equilibrium point has an open region surrounding it which is called the domain of attraction. Liapunov's second method can be used to establish the asymptotic stability of an equilibrium point and to estimate its domain of attraction. For this purpose a function known as Liapunov function is required to be developed. A function V_L is a Liapunov function in the domain Ω if and only if

1. $V_L(x)$ has a continuous derivative. (4.1.11)

$$2. V_L(0) = 0 \quad (4.1.12)$$

$$3. V_L(x) > 0 \text{ for } x \neq 0 \quad (4.1.13)$$

Second and third points refers to a function $V_L(x)$ which is positive definite. Thus a Liapunov function is a continuously differentiable positive definite function of the state. According to Liapunov's second method the equilibrium point $\hat{x} = 0$ in the domain S associated with a constant input is said to be asymptotically stable if along the solution S ,

$$1. \dot{V}_L(x(t)) \leq 0 \quad (4.1.14)$$

$$2. \dot{V}_L(x(t)) \equiv 0 \Rightarrow x(t) \equiv 0 \quad (4.1.15)$$

The advantage of Liapunov's second method which is also called the *direct method* is that it can be used to check the stability (condition 1 and 2) without solving for the solution of the differential equation governing the nonlinear system. But the disadvantage of this method is that it is difficult to obtain a Liapunov function for all physical systems. Also it does not provide any information about the transient response or performance of the system. For example it cannot predict whether a system is over-damped or under-damped or how long it will take to suppress a disturbance. Generally one can take the total energy or the potential energy as the Liapunov function of a system as they are positive definite function.

Example 4.1.2:

Study the stability of the following system by using Liapunov's second method.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 + x_2^3 \end{aligned} \quad (4.1.16)$$

Solution:

For the system the equilibrium point is $\hat{x} = [0 \ 0]^T$ which is obtained by substituting $\dot{x}_1 = \dot{x}_2 = 0$.

Let us consider a Liapunov function

$$V_L(x) = \alpha x_1^2 + \beta x_2^2, \quad \alpha > 0, \beta > 0 \quad (4.1.17)$$

$$\begin{aligned} \dot{V}_L(x) &= \left[\frac{\partial V_L(x)}{\partial x} \right] \dot{x} = [2\alpha x_1 \quad 2\beta x_2] \dot{x} \\ &= [2\alpha x_1 \quad 2\beta x_2] \begin{bmatrix} x_2 \\ -x_1 - x_2 + x_2^3 \end{bmatrix} \\ &= 2\alpha x_1 x_2 + 2\beta x_2 (-x_1 - x_2 + x_2^3) \\ &= 2(\alpha - \beta) x_1 x_2 - 2\beta x_2^2 (1 - x_2^2) \end{aligned} \quad (4.1.18)$$

Since $x_1 x_2$ changes sign at the equilibrium point $x = 0$, so one should constrain α and β so as to eliminate this term. So when $\alpha = \beta$,

$$\dot{V}_L(x) = -2\beta x_2^2(1-x_2^2) \quad (4.1.19)$$

This term $\dot{V}_L(x)$ is negative only when

$$|x_2| < 1 \quad (4.1.20)$$

This satisfies the first condition. To check condition 2, using (4.1.19) one can write

$$\begin{aligned} \dot{V}_L(x) \equiv 0 &\Rightarrow -2\beta x_2^2(1-x_2^2) \equiv 0 \\ &\Rightarrow x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \end{aligned} \quad (4.1.21)$$

Using Eq. (4.1.21) in Eq. (4.1.16), one can obtain $\dot{x}_1 \equiv 0$.

Hence $\dot{V}_L(x) \equiv 0 \Rightarrow x_1 = 0$ and $x_2 \equiv 0$ i.e., $x(t) \equiv 0$. This satisfies the second condition and hence $\hat{x} = [0 \ 0]^T$ is asymptotically stable.

Stability of a first order system

For a dynamic system, one may write the governing differential equation of motion as a set of first order differential equation or one may reduce the governing equation of motion by applying perturbation method to the following form.

$$\dot{x} = F(x, M) \quad (4.1.22)$$

In this equation M represents the control parameters or the system parameters. The steady state response of this system can be obtained by substituting $\dot{x} = 0$, and solving the resulting nonlinear algebraic/transcendental equation. To obtain the stability of the steady state fixed point response, one may perturb or give a small disturbance to the above mentioned equilibrium point and study its behaviour. While for a stable equilibrium point, the system return backs to the original position, for unstable system, after perturbation, the system response grows. Hence, to study stability of the system one uses the following steps.

Considering x_0 as the equilibrium point, substitute $x(t) = x_0 + y(t)$ in equation (4.1.22).

The resulting equation will be

$$\dot{y} = F(x_0 + y, M_0) + D_x F(x_0; M_0)y + O(\|y\|^2) \Rightarrow \dot{y} = D_x F(x_0; M_0)y \equiv Ay \quad (4.1.23)$$

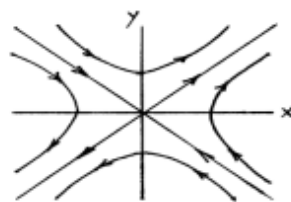
Where $A = \begin{bmatrix} \frac{dF_1}{dx_1} & \frac{dF_1}{dx_2} & \dots & \frac{dF_1}{dx_n} \\ \frac{dF_2}{dx_1} & \frac{dF_2}{dx_2} & \dots & \frac{dF_2}{dx_n} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{dF_n}{dx_1} & \frac{dF_n}{dx_2} & \dots & \frac{dF_n}{dx_n} \end{bmatrix}$

This matrix is known as the Jacobian matrix and the eigenvalues of the constant matrix A provides the information about the local stability of the fixed point x_0 .

Following definitions are required to study the stability and bifurcation of nonlinear systems.

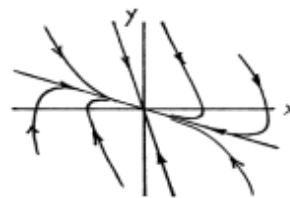
- **Hyperbolic fixed point:** when all of the eigenvalues of A have nonzero real parts it is known as hyperbolic fixed point.
- **Sink:** If all of the eigenvalues of A have negative real part. The sink may be of stable focus if it has nonzero imaginary parts and it is of stable node if it contains only real eigenvalues which are negative.
- **Source:** If one or more eigenvalues of A have positive real part. Here, the system is unstable and it may be of unstable focus or unstable node.
- **Saddle point:** When some of the eigenvalues have positive real parts while the rest of the eigenvalues have negative real parts
- **Marginally stable:** If some of the eigenvalues have negative real parts while the rest of the eigenvalues have zero real parts

Typical saddle, stable node, stable spiral and center are shown in Fig. 4.1.6



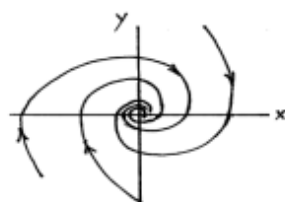
$\ddot{x} - x = 0$

Saddle



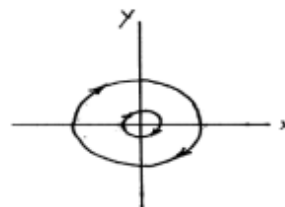
$\ddot{x} + 2\dot{x} + x = 0$

Stable node



$\ddot{x} + \dot{x} + x = 0$

Stable spiral



$\ddot{x} + x = 0$

Center

Fig: 4.1.6: Schematic diagram of trajectory for different equilibrium points (Example 4.1.3).

Example 4.1.3:

Study the equilibrium points for the following dynamic equation (a) $\ddot{x} - x = 0$ (b) $\ddot{x} + x = 0$

(c) $\ddot{x} + \dot{x} + x = 0$ (d) $\ddot{x} + 2\dot{x} + x = 0$.

Solution:

For the system $\ddot{x} - x = 0$, one may write the first order equation by considering

$$\dot{x} = y$$

$$\dot{y} = x$$

Here the equilibrium point is $[0 \ 0]^T$. So the Jacobian matrix can be written as $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the

system has eigenvalue of 1 and -1. Hence according to the definition as the system has a positive real eigenvalues, the equilibrium point is unstable and it is a saddle point. Other solutions are given in the following table.

Original equation	1 st order equation	Equilibrium point	Jacobian matrix	Eigen values	Type of equilibrium point
$\ddot{x} + x = 0$	$\dot{x} = y$ $\dot{y} = -x$	$[0 \ 0]^T$	$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$\lambda_1 = i$ $\lambda_2 = -i$	Center
$\ddot{x} + \dot{x} + x = 0$	$\dot{x} = y$ $\dot{y} = -x - y$	$[0 \ 0]^T$	$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$	$\lambda_1 = (-1 + \sqrt{3}i)/2$ $\lambda_2 = (-1 - \sqrt{3}i)/2$	Stable spiral
$\ddot{x} + 2\dot{x} + x = 0$	$\dot{x} = y$ $\dot{y} = -x - 2y$	$[0 \ 0]^T$	$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$	$\lambda_1 = -1$ $\lambda_2 = -1$	Stable node

Few more theorems related to stability of the nonlinear systems are given below.

Lagrange and Dirichlet Theorem:

According to this theorem, if the potential energy has an isolated minimum at an equilibrium point, the equilibrium state is stable.

Hartman–Grobman theorem or linearization theorem:

It is a theorem about the local behaviour of dynamical systems in the neighborhood of a hyperbolic equilibrium point.

Basically the theorem states that the behaviour of a dynamical system near a hyperbolic equilibrium point is qualitatively the same as the behaviour of its linearization near this equilibrium point provided that no eigenvalue of the linearization has its real part equal to 0.

Lyapunov Theorem:

If the potential energy at an equilibrium point is not a minimum, the equilibrium state is unstable. For a system $\dot{x} = F(x, M)$ with displacement u and potential energy f , and

At saddle point, $\frac{d^2 F}{du^2} = \frac{df}{du} < 0$ and at center $\frac{d^2 F}{du^2} = \frac{df}{du} > 0$

While the motion is unstable near the saddle point it is oscillatory in the neighborhood of center.

Exercise Problems

1. Consider a robotic manipulator whose equation of motion can be written as

$$\tau = M(\theta)\ddot{\theta} + V(\theta, \dot{\theta}) + G(\theta)$$

Study the stability of the system with a control law $\tau = K_p E - K_d \dot{\theta} + G(\theta)$ where $E = \theta_d - \theta$, using Liapunov direct method.

Hints: Liapunov's function $V_L = \frac{1}{2} \dot{\theta}' M(\theta) \dot{\theta} + \frac{1}{2} E' K_p E$.

2. Find the equilibrium point(s) and check their stability for the following system

(a) $\ddot{x} - 4x = 0$ (b) $\ddot{x} + 9x = 0$

(c) $\ddot{x} + \dot{x} + x + x^3 = 0$ (d) $\ddot{x} + 2\dot{x} + x - x^3 = 0$.

References

1. A. H. Nayfeh and D.T. Mook, Nonlinear Oscillation, John Wiley and Sons, Canada, 1979.
2. A. H. Nayfeh and B. Balachandran, Applied Nonlinear Dynamics, John Wiley and Sons, Canada, 1995.
3. S. H. Strogatz, Nonlinear Dynamics and Chaos, Levant Books, First Indian Edition, 2007.

Lecture 2

BIFURCATION ANALYSIS OF FIXED POINT RESPONSE

In nonlinear systems, while plotting the frequency response curves of the system by changing the control parameters, one may encounter the change of stability or change in the number of equilibrium points. These points corresponding to which the number or nature of the equilibrium point changes, are known as bifurcation points. For fixed point response, they may be divided into static or dynamic bifurcation points depending on the nature of the eigenvalues of the system. If the eigenvalues are plotted in a complex plane with their real and imaginary parts (as shown in fig 4.2.1) along X and Y directions, a static bifurcation occurs, if with change in the control parameter, an eigenvalue of the Jacobian matrix crosses the origin of the complex plane. In case of dynamic bifurcation, a pair of complex conjugate eigenvalues crosses the imaginary axis with change in control parameter of the system. Hence, in this case the resulting solution is stable or unstable periodic type.

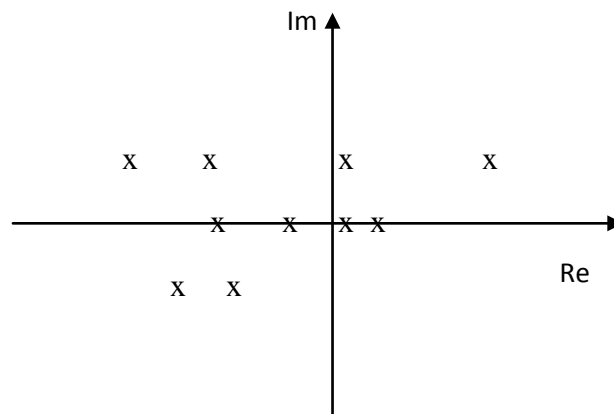


Fig. 4.2.1: Variation of eigenvalues in a complex plane

Static Bifurcation

For a system $\dot{x} = F(x; \mu)$, following two conditions to be satisfied at the static bifurcation point.

$$1. F(x_0; \mu_c) = 0 \quad (4.2.1)$$

2. The Jacobian matrix $\left(D_x F = \frac{\partial F}{\partial x} \right)$ has a zero eigenvalue while all of its other eigenvalues have non zero real parts at $(x_0; \mu_c)$. Hence the point is nonhyperbolic.

There are mainly three different types of static bifurcation points viz., saddle-node, pitch-fork and transcritical bifurcation. One can distinguish saddle-node bifurcation from other bifurcation point by finding $F_\mu = \frac{\partial F}{\partial \mu}$ which is a $n \times 1$ vector of first partial derivatives of the components of F with respect to the control parameter μ and then by constructing a matrix $\begin{bmatrix} D_x F & F_\mu \end{bmatrix}$.

At a saddle-node point F_μ does not belong to the range of $D_x F$ and for pitchfork and transcritical bifurcation point F_μ belongs to the range of $D_x F$. It may be noted that the range of an $n \times n$ matrix A consists of all vectors AZ where $Z \in R^n$. Hence $\begin{bmatrix} D_x F & F_\mu \end{bmatrix}$ has a rank of n at saddle node bifurcation point and rank of $(n-1)$ at other static bifurcation point.

In the state control space all of the branches of fixed points that meet at a saddle-node bifurcation point have the same tangent. For pitchfork and transcritical bifurcation points tangents are not same for all the branches meeting at the bifurcation points.

Saddle-node Bifurcation

The generic form of saddle-node bifurcation is $\dot{x} = F(x; \mu) = \mu + \alpha x^2$. When a scalar control parameter μ is varied, in the $x \sim \mu$ plane, a static bifurcation occurs when the following conditions are satisfied.

$$1. F(x_0; \mu_c) = 0 \text{ i.e., } \mu + \alpha x^2 = 0 \text{ or, } x_0 = \pm \sqrt{\frac{-\mu}{\alpha}} \quad (4.2.2)$$

2. Now the Jacobian matrix $J = D_x F = 2\alpha x$. At the bifurcation point it can be written as

$$J = 2\alpha \sqrt{\frac{-\mu}{\alpha}} = 2\sqrt{-\mu\alpha} \quad (4.2.3)$$

$$\text{Hence, the eigenvalue } \lambda = 2\sqrt{-\mu\alpha} \quad (4.2.4)$$

$$\text{Taking } \alpha = -1, \quad x_0 = \pm\sqrt{\mu} \text{ and } \lambda = -2x_0 \quad (4.2.5)$$

Figure 4.2.2(a) shows the variation of equilibrium point x_0 with control parameter μ for the system $\dot{x} = F(x; \mu) = \mu - x^2$. In this case the equilibrium points are $x = \pm\sqrt{\mu}$ and the eigenvalue is $-2x$ which change its sign at $x_0 = 0$. For positive value of x_0 the response is stable and for negative value of x_0 the response is unstable. From the figure it is clear that the bifurcation point is at $\mu = 0$ and it is a saddle-node bifurcation as the two branches meeting at $\mu = 0$ have same tangent. Also now constructing the $\begin{bmatrix} D_x F & F_\mu \end{bmatrix}$ matrix one can write

$\begin{bmatrix} D_x F & F_\mu \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$, So F_μ does not belongs to range of $D_x F$. Rank of this matrix is 1. So the origin is a saddle-node bifurcation point.

Example 4.2.1:

For a typical dynamic system the frequency-amplitude relation is given by the following equation.

$$\dot{a} = -\zeta a - \bar{\omega}_1^2 \left(\frac{1}{8} \alpha_4 a^2 + \frac{1}{2} \alpha_5 \right) \sin \gamma \tag{4.2.6}$$

$$a \dot{\gamma} = a \sigma - \frac{3}{8} K a^3 - \bar{\omega}_1^2 \left(\frac{3}{8} \alpha_4 a^2 + \frac{1}{2} \alpha_5 \right) \cos \gamma. \tag{4.2.7}$$

Here, $\bar{\omega}_1 = 1 + \epsilon \sigma$. $\zeta, \alpha_4, \alpha_5, k$ are fixed system parameters. The saddle node bifurcation points have been shown in Fig. 4.2.2. (b)

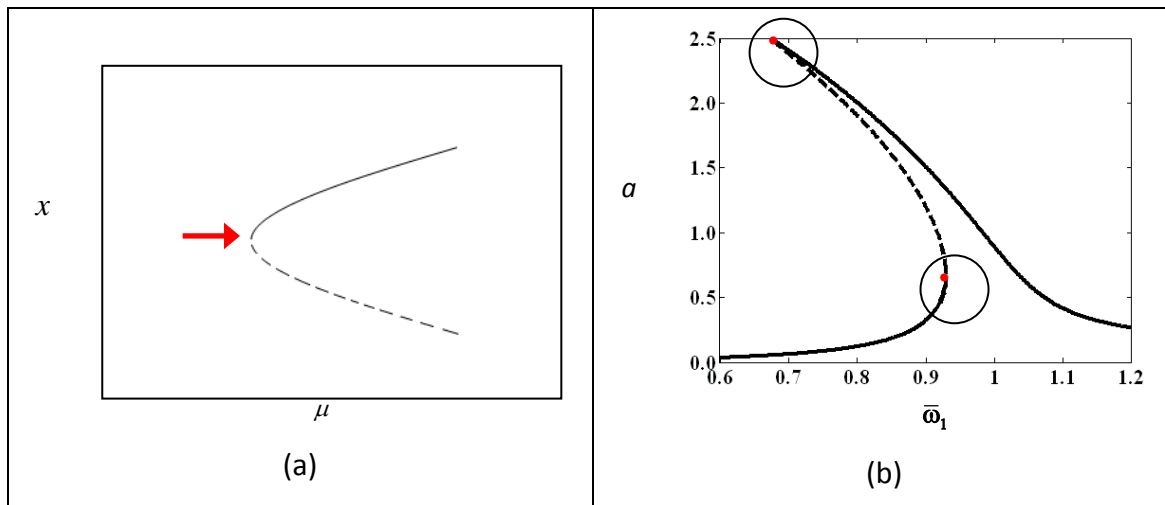


Fig. 4.2.2: Saddle-node bifurcation point corresponding to (a) $\dot{x} = F(x; \mu) = \mu - x^2$, (b) example of a typical second order system.

Pitchfork bifurcation: The normal form for a generic pitchfork bifurcation of a fixed point is

$$\dot{x} = F(x; \mu) = \mu x - x^3 \tag{4.2.8}$$

In this case the equilibrium point is obtained by substituting $F(x; \mu) = 0$.

$$\text{So, } \mu x - x^3 = 0 \Rightarrow x(\mu - x^2) = 0 \tag{4.2.9}$$

So, One can obtain two solutions viz., the trivial solution i.e., $x = 0$ and the non trivial solution $x = \pm\sqrt{\mu}$

The solutions are plotted in Figure 4.2.3.

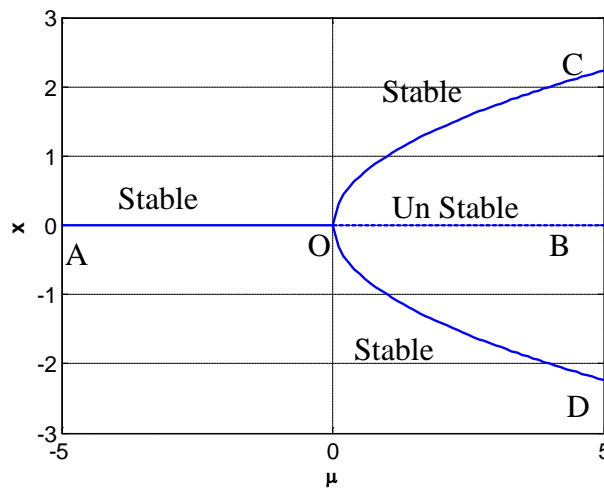


Figure 4.2.3: Super critical pitchfork bifurcation

$$\text{Now } J = D_x F = \mu - 3x^2 \tag{4.2.10}$$

So, for the trivial branch $J = \mu$ and the eigenvalue is $\lambda = \mu$. Hence, the branch AO for which the eigenvalue is negative, is stable and the branch OB which has a positive eigenvalue ($\mu > 0$) is unstable.

Similarly for the nontrivial solution shown by the curve COD, the eigenvalue is

$$\lambda = \mu - 3x^2 = \mu - 3\mu = -2\mu. \tag{4.2.11}$$

As $\mu > 0$ for this branch, the eigenvalue is always negative and so the branch is stable.

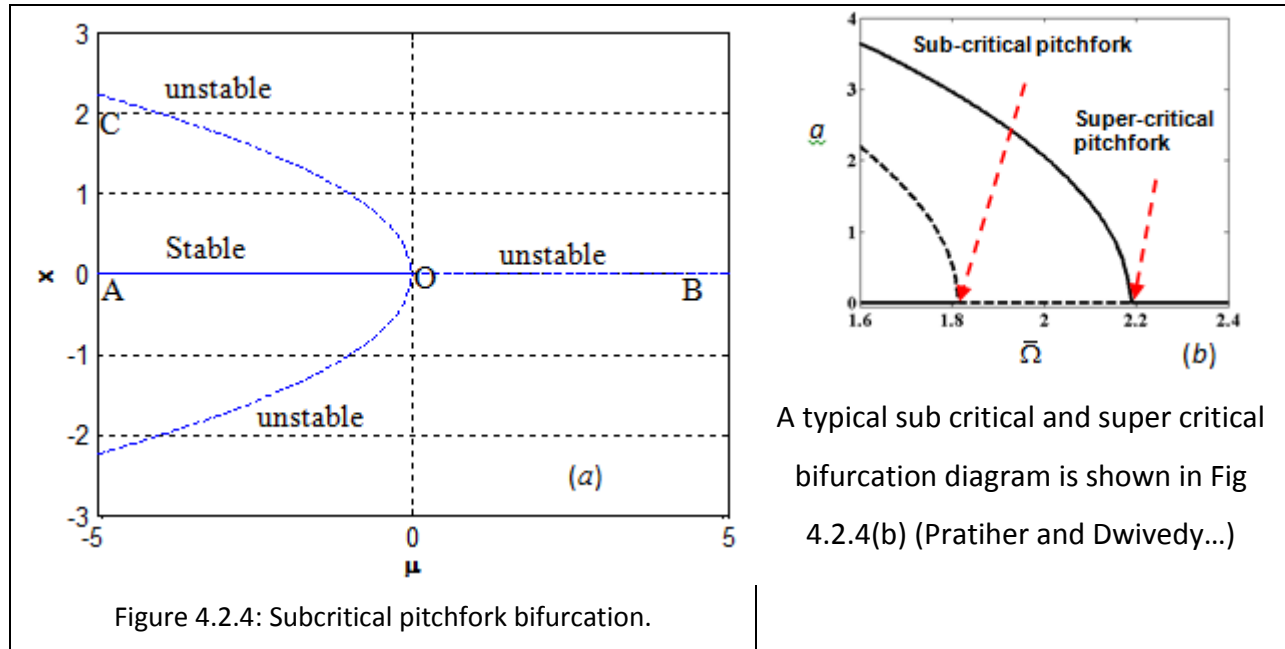
It should be noted that before point ‘O’, the system has only stable trivial state solution and after this point, the system has 3 branches, out of which the trivial state is unstable and non-trivial state is stable. So, there is a change in the number of solution and also, change of stability of the system at point O. Hence this is a bifurcation point and this bifurcation point is known as super-critical pitchfork bifurcation point.

Here, it can be shown that

$$\begin{bmatrix} D_x F & F_\mu \end{bmatrix} = \begin{bmatrix} \mu - 3x^2 & x \end{bmatrix} \quad (4.2.12)$$

At the origin which is a bifurcation point $\begin{bmatrix} D_x F & F_\mu \end{bmatrix} = \begin{bmatrix} \mu - 3x^2 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$. Hence the rank is less than 1.

Now by considering $\dot{x} = F(x; \mu) = \mu x + x^3$, one can find the equilibrium points at $x = 0$ and $x = \pm\sqrt{-\mu}$. Hence, the system has a trivial solution for all values μ and nontrivial solution only for $\mu < 0$ as shown in Figure 4.2.4 (a). In this case, the eigenvalue $\lambda = \mu + 3x^2$. Hence, for trivial state, the branch AO is stable and branch OB is unstable. Similarly for nontrivial state $\lambda = \mu + 3x^2 = \mu - 3\mu = -2\mu \geq 0$ and hence unstable. So, if one increases the control parameter μ from point A, the system will have stable trivial response which will suddenly vanish at point O as there is no stable state beyond this point. Hence, this is a dangerous bifurcation as the system may be subjected to catastrophic failure at this point. This bifurcation point is known as sub-critical pitchfork bifurcation.



A typical sub critical and super critical bifurcation diagram is shown in Fig 4.2.4(b) (Pratiher and Dwivedy...)

Transcritical bifurcation

The normal or generic form of this bifurcation is given by

$$\dot{x} = F(x; \mu) = \mu x - x^2 \tag{4.2.13}$$

So, in this case the equilibrium point is obtained by substituting $F(x; \mu) = 0$ and they are $x = 0$ (trivial state) and $x = \mu$ (non trivial state). Here, the eigenvalue can be given by $\lambda = \mu - 2x$. Figure 4.2.5 shows the variation of x with μ .

Here for branch AO, λ is negative and hence this branch is stable.

Branch OB, λ is positive, hence it is unstable.

Branch CO, $\lambda = \mu - 2x = \mu - 2\mu = -\mu$, (positive), So this branch is unstable

Branch OD, λ is negative and hence this branch is stable.

It may be observed that the trivial and non-trivial branches change their stability at the origin and so the name of this bifurcation point is transcritical bifurcation.

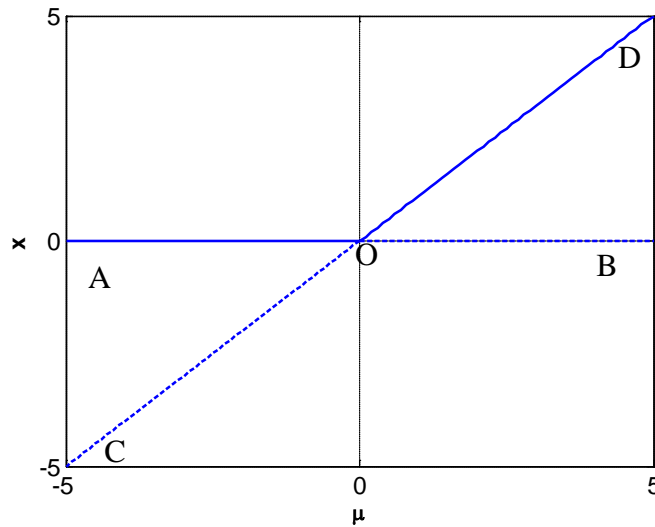


Figure 4.2.5: Variation of x with μ showing transcritical bifurcation at point O.

Hopf bifurcation: For a system $\dot{x} = F(x; \mu)$, following two conditions to be satisfied at the dynamic bifurcation point (Marsden and McCracken 1976, Nayfeh and Balachandran 1995).

1. $F(x_0; \mu_c) = 0$

2. The Jacobian matrix $\left(D_x F = \frac{\partial F}{\partial x} \right)$ has a pair of purely imaginary eigenvalues $\pm i\omega$ while all of its other eigenvalues have non zero real parts at $(x_0; \mu_c)$. Hence, the point is a nonhyperbolic fixed point

3. For $\mu = \mu_c$, let the analytical continuation of the pair of imaginary eigenvalues be $\lambda = \pm i\omega$. Then, $\frac{\partial \lambda}{\partial \mu} \neq 0$ at $\mu = \mu_c$. This condition is known as the transversality condition as the eigenvalue crosses the imaginary axis with nonzero speed.

When all the above three conditions are satisfied, a periodic solution of period $2\pi / \omega$ is developed at $(x_0; \mu_c)$. Such bifurcations are called Hopf bifurcation or Poincare'-Andronov Hopf bifurcation.

The normal form for a generic Hopf bifurcation of a fixed point is given as follows.

$$\dot{x} = \mu x - \omega y + (\alpha x - \beta y)(x^2 + y^2) \tag{4.2.13}$$

$$\dot{y} = \mu x - \omega y + (\alpha x - \beta y)(x^2 + y^2) \quad (4.2.14)$$

One can obtain the fixed point at (0,0) for all values of μ . The Jacobian matrix corresponding to this equilibrium point can be given by

$$A = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix} \quad (4.2.15)$$

Hence, the eigenvalues are obtained by solving the equation $|A - \lambda I| = (\mu - \lambda)^2 + \omega^2 = 0$. So, the eigenvalues are $\lambda_1 = \mu + i\omega$ and $\lambda_2 = \mu - i\omega$. For control parameter $\mu = 0$, a pair of complex conjugate eigenvalues cross the imaginary axis at non-zero values as $\frac{d\lambda_1}{d\mu} = 1$ and $\frac{d\lambda_2}{d\mu} = 1$.

Hence, the three conditions for the Hopf bifurcations are satisfied at the fixed point (0,0). The period of the bifurcating periodic solution at (0,0;0) is $2\pi / \omega$.

In polar form these two equations can be written by substituting $x = r \cos \theta$ and $y = r \sin \theta$ as follows.

$$\dot{r} = \mu r + \alpha r^3, \quad \dot{\theta} = \omega + \beta r^2 \quad (4.2.16)$$

For steady state $\dot{r} = \dot{\theta} = 0$. Substituting $\dot{r} = 0$, it can be observed that the system has both trivial $r = 0$ and non-trivial solution. The trivial solution corresponds to the equilibrium point or the fixed-point response ($x = 0, y = 0$) and the nontrivial fixed point $r \neq 0$ corresponds to a periodic solution with amplitude r and frequency $\dot{\theta}$ which is created due to the Hopf bifurcation.

It may be noted that while supercritical pitchfork and Hopf bifurcation, respectively results in stable fixed-point and periodic responses, the sub-critical pitchfork and Hopf bifurcation, respectively results in unstable fixed-point and periodic responses. Hence, these sub critical bifurcation points are dangerous bifurcation points.

In case of nonlinear vibration many phenomena such as jump-up, jump-down, saturation, multi-stable region along with different types of responses such as fixed-point, periodic, quasi-periodic and chaotic are observed. Many bifurcation phenomena such as sub and super critical pitchfork, Hopf, saddle-point, period doubling etc are observed. One may observe different type of crisis phenomena in chaotically modulated system.

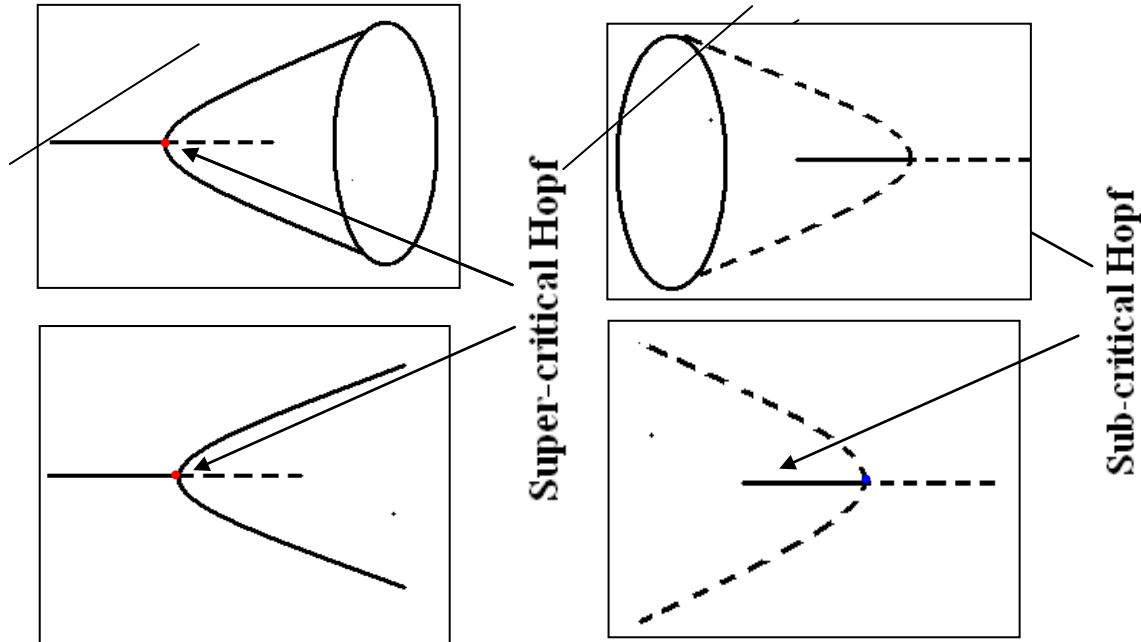


Fig. 4.2.6 (a) super-critical Hopf bifurcation, (b) sub-critical Hopf bifurcation.

Example 4.2.4: Stability analysis of multi-degree of freedom system

The reduced equations for a based excited cantilever beam with attached mass can be given by

$$2\omega_1(\xi_1 a_1 + a_1') - \frac{1}{2}\{f_{11}a_1 \sin 2\gamma_1 + f_{12}a_2 \sin(\gamma_1 - \gamma_2)\} + 0.25Q_{12}a_2 a_1^2 \sin(3\gamma_1 - \gamma_2) = 0 \quad (4.2.17)$$

$$2\omega_1 a_1 \left(\gamma_1' - \frac{1}{2}\sigma_1 \right) - \frac{1}{2}\{f_{11}a_1 \cos 2\gamma_1 + f_{12}a_2 \cos(\gamma_1 - \gamma_2)\} + \frac{1}{4} \sum_{j=1}^2 \alpha_{e1j} a_j^2 a_1 + \frac{1}{4} Q_{12} a_2 a_1^2 \cos(3\gamma_1 - \gamma_2) = 0 \quad (4.2.18)$$

$$2\omega_2(\xi_2 a_2 + a_2') - \frac{1}{2}\{f_{21}a_1 \sin(\gamma_2 - \gamma_1)\} + \frac{1}{4} Q_{21} a_1^3 \sin(\gamma_2 - 3\gamma_1) = 0 \quad (4.2.19)$$

$$2\omega_2 a_2 (\gamma_2' + \sigma_2 - 1.5\sigma_1) - \frac{1}{2} f_{21} a_1 \cos(\gamma_2 - \gamma_1) + \frac{1}{4} \sum_{j=1}^2 \alpha_{e2j} a_j^2 a_2 + \frac{1}{4} Q_{21} a_1^3 \cos(\gamma_2 - 3\gamma_1) = 0 \quad (4.2.20)$$

For steady state, $a_1' = \gamma_1' = a_2' = \gamma_2' = 0$. These non-linear algebraic equations can be solved numerically to obtain the fixed point response of the system.

By directly perturbing the reduced equations, one can study the stability of the non-trivial steady state solution. But, as the reduced Eqs. (4.2.17-20) have the coupled terms $a_1\dot{\gamma}_1$ and $a_2\dot{\gamma}_2$, the perturbed equations will not contain the perturbations $\Delta\dot{\gamma}_1$ or $\Delta\dot{\gamma}_2$ for trivial solutions and hence the stability of the trivial state cannot be studied by directly perturbing these equations. To circumvent this difficulty, normalization method can be adopted by introducing the transformation

$$p_i = a_i \cos \gamma_i, \quad q_i = a_i \sin \gamma_i, \quad i = 1, 2 \quad (4.2.21)$$

into the reduced equations (4.2.17-4.2.20). Carrying out trigonometric manipulations, one arrives at the following normalized reduced equations or the Cartesian form of modulation equations:

$$\begin{aligned} & 2\omega_1(p_1' + \xi_1 p_1) + \left(\omega_1 \sigma_1 - \frac{1}{2} f_{11} \right) q_1 + \frac{1}{2} f_{12} q_2 \\ & + \frac{1}{4} Q_{12} \{ q_2 (q_1^2 - p_1^2) + 2p_1 p_2 q_1 \} - \frac{1}{4} \sum_{j=1}^2 \alpha_{e1j} q_1 (p_j^2 + q_j^2) = 0 \end{aligned} \quad (4.2.22)$$

$$\begin{aligned} & 2\omega_1(q_1' + \xi_1 q_1) + \left(\omega_1 \sigma_1 - \frac{1}{2} f_{11} \right) p_1 + \frac{1}{2} f_{12} p_2 \\ & + \frac{1}{4} Q_{12} \{ p_2 (p_1^2 - q_1^2) + 2p_1 q_1 q_2 \} - \frac{1}{4} \sum_{j=1}^2 \alpha_{e1j} p_1 (p_j^2 + q_j^2) = 0 \end{aligned} \quad (4.2.23)$$

$$\begin{aligned} & 2\omega_2(p_2' + \xi_2 p_2) + \frac{1}{2} f_{21} q_1 + \omega_2 (3\sigma_1 - 2\sigma_2) q_2 \\ & - \frac{1}{4} Q_{21} q_1 (3p_1^2 - q_1^2) - \frac{1}{4} \sum_{j=1}^2 \alpha_{e2j} q_2 (p_j^2 - q_j^2) = 0 \end{aligned} \quad (4.2.24)$$

$$\begin{aligned} & 2\omega_2(q_2' + \xi_2 q_2) - \frac{1}{2} f_{21} p_1 - \omega_2 (3\sigma_1 - 2\sigma_2) p_2 \\ & + \frac{1}{4} Q_{21} p_1 (p_1^2 - 3q_1^2) + \frac{1}{4} \sum_{j=1}^2 \alpha_{e2j} p_2 (p_j^2 + q_j^2) = 0 \end{aligned} \quad (4.2.25)$$

Now perturbing the above equations, one obtains

$$\{ \Delta p_1', \Delta q_1', \Delta p_2', \Delta q_2' \}^T = [J_c] \{ \Delta p_1, \Delta q_1, \Delta p_2, \Delta q_2 \}^T \quad (4.2.26)$$

where T is the transpose and $[J_c]$ is the Jacobian matrix whose eigenvalues will determine the stability and bifurcation of the system. The stability boundary for the linear system (i.e. the trivial

state) can be obtained from the eigenvalues of the matrix $[J_c]$ by letting $p_1 = p_2 = q_1 = q_2 = 0$. A detailed numerical results can be found in the work of Kar and Dwivedy (1999).

Exercise Problems

1. Study the stability of the following one dimensional systems by plotting the stable and unstable branches. Also, write about the type of bifurcation observed in the system.

(a) $\dot{x} = \mu - 4x^2$ (b) $\dot{x} = \mu + 4x^2$ (c) $\dot{x} = \mu x + 4x^2$ (d) $\dot{x} = \mu x - 4x^2$ (e) $\dot{x} = \mu x + 4x^3$ (f) $\dot{x} = \mu x - 4x^3$

2. Develop the equation for steady state solution and the Jacobian matrix from the reduced equations obtained by solving the Duffing equation using method of multiple scales as given below. Plot the frequency response curve with stable and unstable branches with different colours.

$$a' = -\mu a + \frac{1}{2} \frac{f}{\omega_0} \sin \gamma$$

$$a\gamma' = \sigma a - \frac{3}{8} \frac{\alpha}{\omega_0} a^3 + \frac{1}{2} \frac{f}{\omega_0} \cos \gamma$$

Answer: Equation for frequency response is given by

$$\sigma = \frac{3}{8} \frac{\alpha}{\omega_0} a^2 \pm \left(\frac{f^2}{4\omega_0^2 a^2} - \mu^2 \right)^{\frac{1}{2}}$$

The Jacobian matrix is

$$J = \begin{vmatrix} -\mu & -a_0 \left(\sigma - \frac{3\alpha a_0^2}{8\omega_0} \right) \\ \frac{1}{a_0} \left(\sigma - \frac{9\alpha a_0^2}{8\omega_0} \right) & -\mu \end{vmatrix}$$

3. Find the Jacobian matrix for the system given in equations (4.2.6) and (4.2.7).

4. Find the Jacobian matrix for the system given by the following equations.

$$\dot{a} = -\zeta a + \frac{k_1}{8} a^3 + \frac{f_1}{4} a \sin \gamma,$$

$$a\dot{\gamma} = 2a\left(\frac{\bar{\omega}-1}{\varepsilon}\right) - \frac{3}{4}\left(\alpha_1 - \alpha_2 + \frac{\alpha_3}{3}\right)a^3 + \frac{1}{4}a^3k_1 \sin \gamma + \frac{f_1}{2}a \cos \gamma.$$

Answer: The Jacobian matrix is given by

$$J = \begin{bmatrix} -\zeta + \frac{3k_1}{8}a_0^2 + \frac{f_1}{4}\sin \gamma_0 & \frac{f_1}{4}a_0 \cos \gamma_0 \\ -\frac{3}{2}\left(\alpha_1 - \alpha_2 + \frac{\alpha_3}{3}\right)a_0 + \frac{1}{2}a_0k_1 \sin \gamma_0 & \frac{1}{4}a_0^2k_1 \cos \gamma_0 - \frac{f_1}{2}\sin \gamma_0 \end{bmatrix}$$

5. Write a Matlab code to plot time response and phase portrait of a periodic response (a) $y = 5 \sin 5t$ (b) $y = 5(\sin 5t + \sin 10t)$.

Solution

```
t=0:0.01:10;
y=5*sin(5*t);
yt=25*cos(5*t);
y1=5*(sin(5*t)+sin(10*t));
yt1=5*(5*cos(5*t)+10*cos(10*t));
figure(1)
subplot(2,2,1) plot(t,y)
xlabel('t')
ylabel('y')
subplot(2,2,2)
plot(y,yt)
xlabel('t')
ylabel('y')
subplot(2,2,3)
plot(t,y1)
xlabel('t')
ylabel('y1')
subplot(2,2,4) plot(y1,yt1)
xlabel('y') ylabel('yt1')
```

Reference:

1. A. H. Nayfeh, and B. Balachandran, *Applied Nonlinear Dynamics*, Wiley, 1995.

2. R. C. Kar and S. K. Dwivedy, Non-linear dynamics of a slender beam carrying a lumped mass with principal parametric and internal resonances. International Journal of Nonlinear Mechanics, 34, (3), 515-529, 1999.
3. J.E. Marsden and M. McCracken, The Hopf Bifurcation and its Applications, Springer-Verlag, New York, 1976

Module 4 Lecture 3

Stability analysis of Periodic response

In this lecture stability analysis of periodic response will be discussed. As pointed out in the previous lecture, after Hopf bifurcation, a periodic response occurs in the system. Also, in many resonance conditions, the system yields a periodic response. Hence, one should study the stability and bifurcation of the periodic response.

Let us consider the forced Duffing equation as given below.

$$\ddot{u} + \omega_0^2 u = -2\varepsilon\mu\dot{u} - \varepsilon\alpha u^3 + K \cos \Omega t \quad (4.3.1)$$

Using Method of multiple scales, the solution of this equation for the super harmonic resonance condition ($\omega_0 \approx 3\Omega$) can be written (Nayfeh and Mook 1979) as

$$u = a \cos(3\Omega t - \gamma) + 2\Lambda \cos \Omega t + o(\varepsilon). \quad (4.3.2)$$

Here a and γ can be obtained from the following reduced equations.

$$\begin{aligned} a' &= -\mu a - \frac{\alpha\Lambda^3}{\omega_0} \sin \gamma \\ a\gamma' &= \left(\sigma - \frac{3\alpha\Lambda^2}{\omega_0}\right)a - \frac{3\alpha}{8\omega_0} a^3 - \frac{\alpha\Lambda^3}{\omega_0} \cos \gamma \end{aligned} \quad (4.3.3)$$

For steady state as a' and γ' are zero, Eq. (4.3.3) can be written as

$$\begin{aligned} -\mu a_0 - \frac{\alpha\Lambda^3}{\omega_0} \sin \gamma_0 &= 0 \\ \left(\sigma - \frac{3\alpha\Lambda^2}{\omega_0}\right)a_0 - \frac{3\alpha}{8\omega_0} a_0^3 - \frac{\alpha\Lambda^3}{\omega_0} \cos \gamma_0 &= 0 \end{aligned} \quad (4.3.4)$$

Here a_0 and γ_0 are the steady state solutions. Now eliminating γ_0 from the above equations, one may obtain the following equation for the frequency response curve

$$\mu^2 a_0^2 + a_0^2 \left(\sigma - \frac{3\alpha\Lambda^2}{\omega_0} - \frac{3\alpha}{8\omega_0} a_0^3 \right) = \frac{\alpha^2 \Lambda^6}{\omega_0^2} \quad (4.3.5)$$

It may be noted at this stage that though the actual solution u is periodic (refer Eq. 4.3.2), one may study the stability of the fixed point response a_0 as discussed in the previous lecture by perturbing the solution as given below.

$$a = a_0 + a_1, \gamma = \gamma_0 + \gamma_1 \quad (4.3.6)$$

Substituting Eq. (4.3.6) in the reduced equations (4.3.3) one can obtain the following sets of equations.

$$\begin{aligned} a' &= -\mu a_1 - \frac{\alpha\Lambda^3}{\omega_0} \gamma_1 \cos \gamma_0 \\ \gamma_1' &= -\frac{3\alpha}{8\omega_0} a_0 a_1 + \frac{\alpha\Lambda^3}{\omega_0 a_0^2} a_1 \cos \gamma_0 + \frac{\alpha\Lambda^3}{\omega_0 a_0} \gamma_1 \sin \gamma_0 \end{aligned} \quad (4.3.7)$$

The Jacobian matrix can be given by

$$\mathbf{J} = \begin{bmatrix} \mu & \frac{\alpha\Lambda^3}{\omega_0} \cos \gamma_0 \\ \frac{3\alpha a_0}{8\omega_0} - \frac{\alpha\Lambda^3}{\omega_0} \cos \gamma_0 & \alpha - \frac{\alpha\Lambda^3}{\omega_0 a_0} \sin \gamma_0 \end{bmatrix} \quad (4.3.8)$$

Now one can study the stability by finding the eigenvalues λ of the Jacobian matrix. This can be obtained by finding the determinant of $|\mathbf{J} - \lambda \mathbf{I}|$ which yields the following equation.

$$(\lambda + \mu)^2 = \frac{\alpha\Lambda^3}{\omega_0} \cos \gamma_0 \left(\frac{\alpha\Lambda^3}{\omega_0 a_0^2} \cos \gamma_0 - \frac{3\alpha a_0}{8\omega_0} \right) \quad (4.3.9)$$

The solution is stable if the real parts of the eigenvalues are negative and it is unstable if at least one of the real parts of the eigenvalues is positive. So in this way one can study the stability of the periodic solution by studying the corresponding solution of the fixed point response.

Instead of using perturbation methods to obtain the steady state response of the solution, if one uses the harmonic balance method the solution can be given by

$$u = u_0 = A_1 \cos \Omega t + B_1 \sin \Omega t + A_3 \cos 3\Omega t + B_3 \sin 3\Omega t \quad (4.3.10)$$

The constants of this equation (4.3.10) can be obtained by following the standard procedure. Now to study the stability of the periodic solution (4.3.10) one has to perturb the original equation. In this case, letting $u = u_0 + u_1$ in Eq. (4.3.1) one can write

$$\ddot{u}_1 + \omega_0^2 u_1 + 2\varepsilon\mu\dot{u}_1 + 3\varepsilon\alpha[A_1 \cos \Omega t + B_1 \sin \Omega t + A_3 \cos 3\Omega t + B_3 \sin 3\Omega t]^2 u_1 = 0 \quad (4.3.11)$$

$$\text{Or, } \ddot{u}_1 + \omega_0^2 u_1 + 2\varepsilon\mu\dot{u}_1 + \varepsilon f(t)u_1 = 0 \quad (4.3.12)$$

where $f(t)$ is a periodic function of time. Equation (4.3.12) is in the form of Mathieu-Hill's type of equation whose solution can be obtained by using Floquet theory.

FLOQUET THEORY

It is used for characterizing the functional behaviour of parametrically excited systems which are represented by partial differential equation with periodic coefficients (Nayfeh and Mook 1979).

Let us consider the equation

$$\ddot{u} + p_1(t)\dot{u} + p_2(t)u = 0 \quad (4.3.13)$$

Since Eq. (4.3.13) is a linear second order homogeneous differential equation, there exist two linear non zero independent fundamental sets of solutions $u_1(t)$ and $u_2(t)$.

$$\text{Hence, one can write } u(t) = c_1 u_1(t) + c_2 u_2(t) \quad (4.3.14)$$

Where c_1 and c_2 are constants. Since, $p_1(t) = p_1(t+T)$,

$$\ddot{u}(t+T) = -p_1(t+T)\dot{u}(t+T) - p_2(t+T)u(t+T) = -p_1(t)\dot{u}(t+T) + p_2(t)u(t+T)$$

Hence, if $u_1(t)$ and $u_2(t)$ are fundamental set of solution of Eq. (4.3.13), $u_1(t+T)$ and $u_2(t+T)$ are also a fundamental set of solutions of Eq. (4.3.13). So one can write

$$\begin{aligned} u_1(t+T) &= a_{11}u_1(t) + a_{12}u_2(t) \\ u_2(t+T) &= a_{21}u_1(t) + a_{22}u_2(t) \end{aligned} \quad (4.3.15)$$

Where a_{ij} are the elements of a constant nonsingular matrix $[A]$. This matrix is not unique and depends on the fundamental sets being used. There exists a fundamental set of solutions for which the off diagonal terms of the matrix $[A]$ are zero. Hence, in this case one may write

$$\begin{aligned} v_1(t+T) &= a_{11}v_1(t) = \lambda_1 v_1(t) \\ v_2(t+T) &= a_{22}v_2(t) = \lambda_2 v_2(t). \end{aligned} \quad (4.3.16)$$

Here λ is a constant which may be complex. These solutions are called normal or Floquet solutions

Hence, one can write

$$v_i(t+T) = \lambda_i v_i(t), \quad i = 1, 2 \quad (4.3.17)$$

$$\text{So, } v_i(t+2T) = v_i((t+T)+T) = \lambda_i v_i(t+T) = \lambda_i \lambda_i v_i(t) = \lambda_i^2 v_i(t) \quad (4.3.18)$$

$$\text{Similarly, } v_i(t+nT) = \lambda_i^n v_i(t) \quad (4.3.19)$$

Here n is an integer. For the steady state as time tends to ∞ , n should tends to ∞ . Hence, for steady state

$$v_i(t) = \begin{cases} 0 & \text{if } |\lambda_i| < 1 \\ \infty & \text{if } |\lambda_i| > 1 \end{cases} \quad (4.3.20)$$

When $\lambda_i = 1$, $v_i(t)$ is periodic with period T and when $\lambda_i = -1$, $v_i(t)$ is periodic with period $2T$. This forms the basis of the bifurcation analysis of periodic response. The system is stable if λ_i 's remain within the unit circle, and are unstable if they are out of the unit circle. On the boundary of the unit circle the solution may be periodic or two periodic depending on positive or negative values of λ_i .

Now multiplying $\exp[-\gamma_i(t+T)]$ in Eq. (4.3.17) one can write

$$\exp[-\gamma_i(t+T)]v_i(t+T) = \lambda_i \exp(-\gamma_i T) \exp(-\gamma_i t)v_i(t) \quad (4.3.21)$$

Now by choosing

$$\lambda_i = \exp(\gamma_i T), \quad (4.3.22)$$

Eq. (4.3.21) can be written as

$$\exp[-\gamma_i(t+T)]v_i(t+T) = \exp(-\gamma_i t)v_i(t) = \phi_i \quad (4.3.23)$$

From Eq. (4.3.22) $\phi_i = \exp(-\gamma_i t)v_i(t)$ is a periodic function with period T . So Eq. (4.3.16) can be written as

$$\begin{aligned} v_1(t) &= \exp(\gamma_1 t)\phi_1(t) \\ v_2(t) &= \exp(\gamma_2 t)\phi_2(t) \end{aligned} \quad (4.3.24)$$

Where, $\phi_i(t+T) = \phi_i(t)$.

From Eq. (4.3.22) one can write

$$\gamma_i = \frac{1}{T} \ln(\lambda_i) \quad (4.3.25)$$

Corresponding to λ_i greater than 1, γ_i is positive and for λ_i less than 1, γ_i is negative. So for a stable system γ_i should be negative.

Hence, either by finding λ_i or by finding γ_i , the stability of the steady state solution can be determined.

Example 4.3.1

Use Floquet theory to study the stability of the following equation.

$$\ddot{u} + p_1(t)\dot{u} + p_2(t)u = 0 \tag{4.3.26}$$

With initial condition

$$u_1(0) = 1, \quad \dot{u}_1(0) = 0, \quad u_2(0) = 0, \quad \dot{u}_2(0) = 1 \tag{4.3.27a}$$

and after one cycle $T = 1$ sec. let

$$u_1(T) = 4, \quad \dot{u}_1(T) = 0.5, \quad u_2(T) = 2, \quad \dot{u}_2(T) = 2 \tag{4.3.27b}$$

Solution

Using the initial conditions (4.3.27 b) in the following equations

$$\begin{aligned} u_1(t+T) &= a_{11}u_1(t) + a_{12}u_2(t) \\ u_2(t+T) &= a_{21}u_1(t) + a_{22}u_2(t) \end{aligned} \tag{4.3.28}$$

$$\begin{aligned} \dot{u}_1(t+T) &= a_{11}\dot{u}_1(t) + a_{12}\dot{u}_2(t) \\ \dot{u}_2(t+T) &= a_{21}\dot{u}_1(t) + a_{22}\dot{u}_2(t) \end{aligned} \tag{4.3.29}$$

one can obtain

$$a_{11} = u_1(T), \quad a_{21} = u_2(T), \quad a_{12} = \dot{u}_1(T), \quad a_{22} = \dot{u}_2(T) \tag{4.3.30}$$

$$\text{Or } A = \begin{bmatrix} u_1(T) & \dot{u}_1(T) \\ u_2(T) & \dot{u}_2(T) \end{bmatrix} \tag{4.3.31}$$

Finding the determinant of $A - \lambda I$ matrix one may write

$$\lambda^2 - 2\alpha\lambda + \Delta = 0 \tag{4.3.32}$$

where

$$\alpha = \frac{1}{2}[u_1(T) + \dot{u}_2(T)], \Delta = u_1(T)\dot{u}_2(T) - \dot{u}_1(T)u_2(T) \quad (4.3.33)$$

The parameter Δ is known as the Wronskian determinant of $u_1(T)$ and $u_2(T)$.

So in the present case, $\alpha = 3$, $\Delta = 2 - 1 = 1$ and hence

$$\lambda^2 - 6\lambda + 1 = 0$$

$$\text{Hence, } \lambda = 3 \pm \frac{1}{2}\sqrt{36 - 4} \Rightarrow \lambda = 5.828 \text{ and } 0.1715.$$

As one of the λ value is outside the unit circle, the system is unstable. Also, one may find γ by using Eq. (4.3.25). As one of the γ is positive the system is unstable.

Exercise problem

Problem 4.3.1: Study the stability of Hill's equation by using Floquet theory.

Problem 4.3.2: Study the stability of Mathieu's equation by using Floquet theory.

Problem 4.3.3: Use Floquet theory to study the stability of the following equation.

$$\ddot{u} + p_1(t)\dot{u} + p_2(t)u = 0$$

With initial condition

$$u_1(0) = 1, \quad \dot{u}_1(0) = 0, \quad u_2(0) = 0, \quad \dot{u}_2(0) = 1$$

and after one cycle $T = 1$ sec. let

$$u_1(T) = -6, \quad \dot{u}_1(T) = 3, \quad u_2(T) = 1, \quad \dot{u}_2(T) = 2$$

References:

A. H. Nayfeh and D. T. Mook, Nonlinear Oscillations, Wiley, 1979

Module 4 Lecture 4**Limit cycles and Bifurcation of Periodic Response**

In this lecture a special case of periodic response i.e., the limit cycles is briefly discussed. Also, the bifurcation analysis of the periodic response has been described. As discussed in the previous lecture, it may be noted that a solution is said to be periodic if it repeats with certain time period T . Hence, for a periodic solution $x=X(t)$,

$$X(t+T) = X(t) \quad (4.4.1)$$

So, for a periodic solution given by Eq. (4.4.1) it should have the following properties.

- **Minimum period T**
- **Form a closed orbit in phase portrait.**
- **Could be treated as a fixed point in Poincare' section**

Figure 4.4.1 (a) shows a periodic response $u = a \cos(\omega t + \beta)$ where $a = 3, \omega = 1, \beta = -3.15$. The phase portrait is shown in Fig. 4.4.1(b)

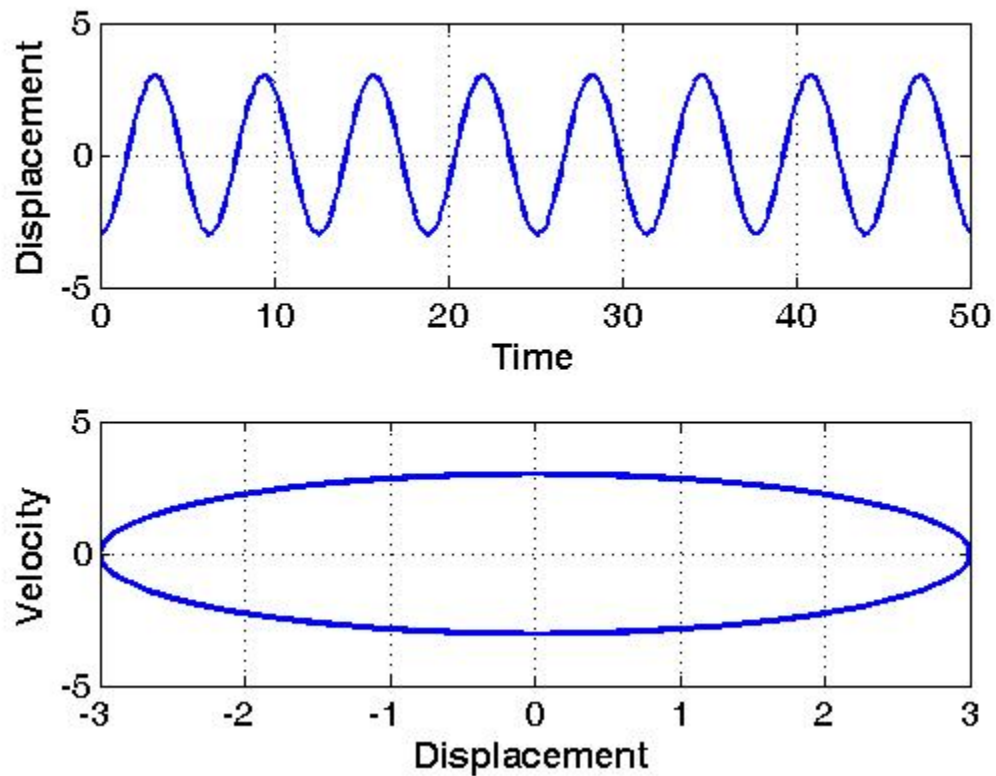


Fig. 4.4.1: (a)Time response (b) Phase portrait

- Limit Cycle: A periodic solution is said to be limit cycle if there is no other periodic solutions sufficiently close to it.
- A limit cycle is an isolated periodic solution and corresponds to an isolated closed orbit in the state space
- Every trajectory initiated near a limit cycle approaches it either as $t \rightarrow \infty$ or $t \rightarrow -\infty$. For example Figure 4.4.2 and 4.4.3 shows the phase portrait and time response for the Van der pol's equation with two different initial condition. The Van der pol's equation is given by

$$\ddot{x} + x - \lambda(1 - x^2)\dot{x} = 0. \quad (4.4.2)$$

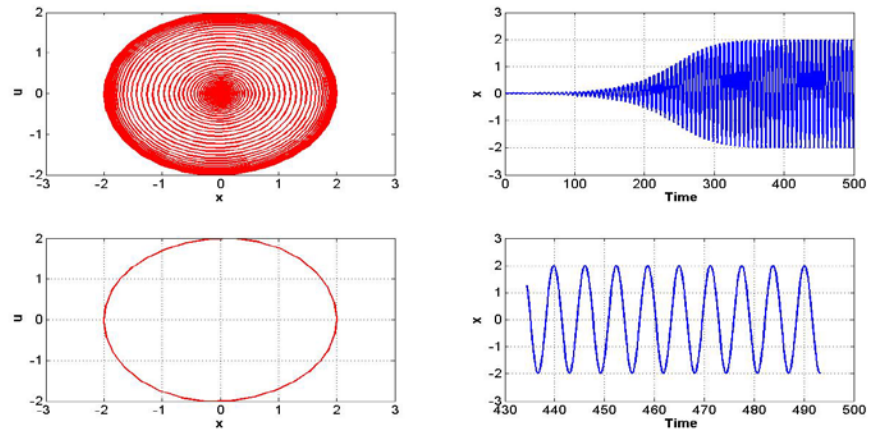


Fig. 4.4.2: (a) Phase portrait, (b) time response, (c) steady state phase portrait and (d) steady state time response with initial point (0.001,0.001) with $\lambda = xxx$.

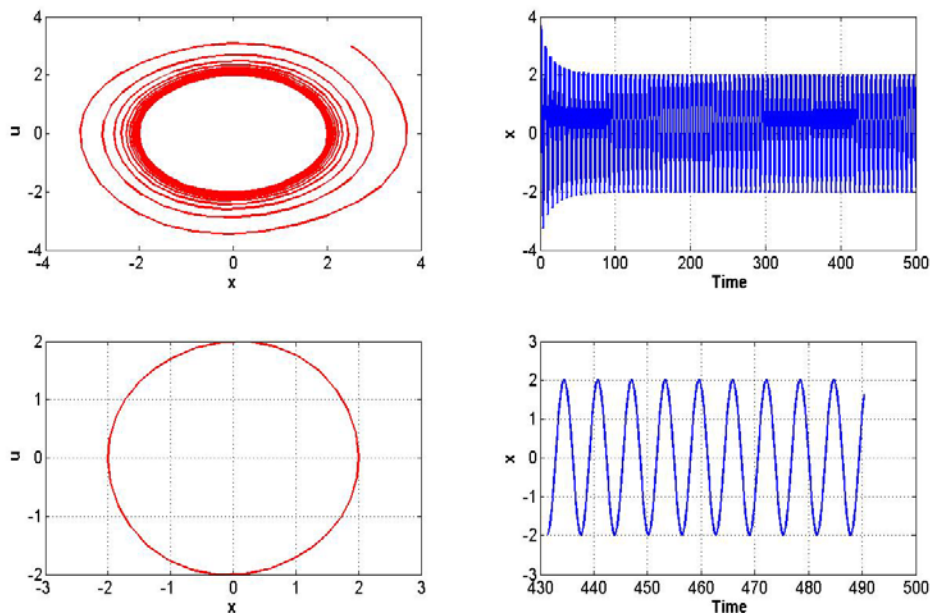


Fig. 4.4.3: (a) Phase portrait, (b) time response, (c) steady state phase portrait and (d) steady state time response with initial point (2.5,3.0). $\lambda = ?$

While in Figure 4.4.2 the initial point is taken within the limit cycle in Figure 4.4.3 the initial point is taken outside the limit cycle. With increase in time it is clearly observed that both the initial points lead to the same limit cycle. It may be observed from Eq.4.4.2 that when $|x| > 1$ $|x| < 1$, for positive λ , the system has a negative damping which makes the system to grow with time. Similarly, for $|x| > 1$

Existence of closed orbits

To rule out existence of closed orbits following rules/theorems may be used.

1. Closed orbits are impossible in gradient systems.

A system which can be written in the form $\dot{x} = -\nabla V(x)$ for some continuously differentiable, single valued scalar function $V(x)$ is called a gradient function.

Let $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$ be a smooth vector field defined on phase plane. For this system to be a gradient system $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$

2. Dulac’s criterion: Let $\dot{x} = F(x)$ be a continuously differentiable vector field defined on a simply connected subset R of the plane. If there exists a continuously differentiable real valued function $g(x)$ such that $\nabla \cdot (g(\dot{x}))$ has one sign throughout R , then there is no closed orbit laying entirely in R .

3. A system for which a Liapunov function can be constructed will have no closed orbits.

Relaxation oscillation

Let us again consider the van der pol’s oscillator $\ddot{x} + x - \lambda(1 - x^2)\dot{x} = 0$ when λ greater than 1.

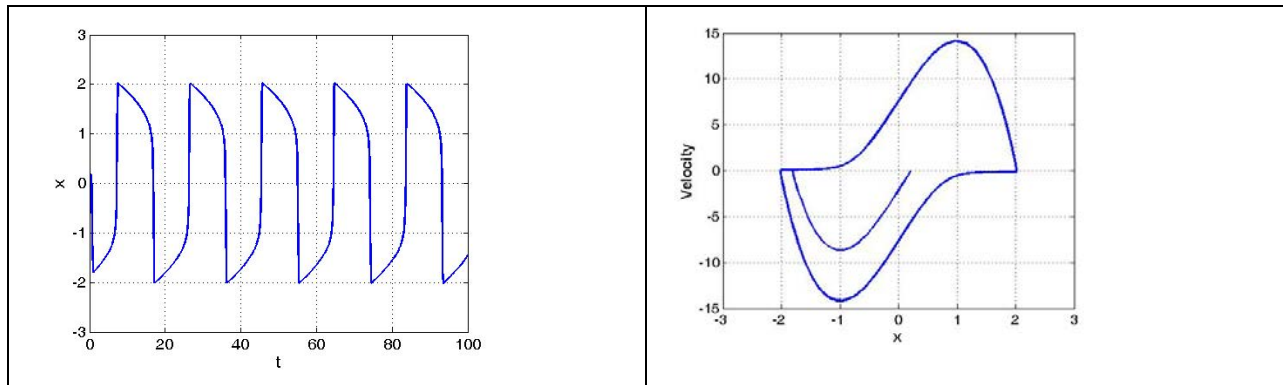


Fig4.4.4(a) Time response, (b) phase portrait for vander pol’s oscillator with $\lambda=10$,with initial condition (0.2,0) .

Figure 4.4.4 shows the time response and phase portrait for this oscillator when $\lambda=10$, with initial condition (0.2,0). By changing the initial condition to (2,0) Figure 4.4.5 shows the time response, velocity response and phase portrait for this oscillator when $\lambda=10$. It can be observed from these two plots that unlike the periodic response shown in Fig. 4.4.1, here, the system has two widely separated time scales one for slow motion and one for fast motion as evident from the velocity response. This type of

oscillation is known as relaxation oscillation as the system relaxes or moves with very slow speed followed by a jump in the speed in a particular cycle.

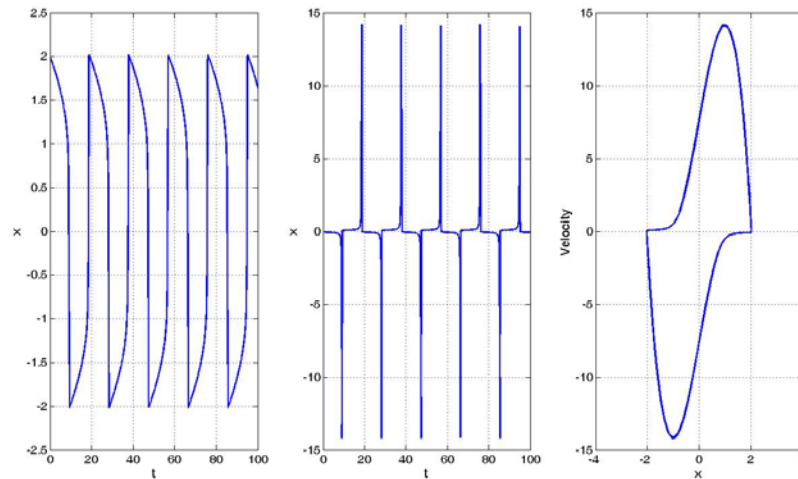


Fig 4.4.3(a) Time response, (b) velocity response, (c) phase portrait for van der pol's oscillator with $\lambda = 10$, with initial condition (2,0) .

```
% Matlab code Relaxation oscillation (van der pol's eqn)
clc
clear all
global mu

mu=10;

[T,Y] = ode45(@vdp_r,[0 30],[2 0]);

subplot(1,3,1)
%figure(1)

plot(T,Y(:,1),'-')

set(findobj(gca,'Type','line'),'Color','b','LineWidth',2);
set(gca,'FontSize',14)
xlabel('t','fontSize',14,'fontWeight','b');
ylabel('x','fontSize',14,'fontWeight','b');
grid on

subplot(1,3,3)
%figure(2)
plot(Y(:,1),Y(:,2),'-');

set(findobj(gca,'Type','line'),'Color','b','LineWidth',2);
set(gca,'FontSize',14)
```

```

xlabel('x','fontsize',14,'fontweight','b');
ylabel('Velocity','fontsize',14,'fontweight','b');

subplot(1,3,2)
%figure(3)

plot(T,Y(:,2),'-');

set(findobj(gca,'Type','line'),'Color','b','LineWidth',2);
set(gca,'FontSize',14)
xlabel('t','fontsize',14,'fontweight','b');
ylabel('x','fontsize',14,'fontweight','b');
grid on

function dz = vdp_r(t,z)
%vander pol's oscillators: relaxation oscillation
global mu

dz = zeros(2,1);    % a column vector

dz(1) = z(2);
dz(2) = -(mu*(z(1)^2-1).*z(2))-z(1);

```

Bifurcation Analysis of Periodic Response

Similar to the bifurcation to the fixed point response, here also, a bifurcation in the periodic response occurs if by changing the control parameter one observes a qualitative or quantitative change in the system response branches. A bifurcation that requires at least m independent control parameters to occur is called a codimension- m bifurcation.

As discussed in the previous lecture, a periodic response will be stable if all the eigenvalues of the monodromy matrix (Floquet multiplier) lies inside the unit Circle which is shown in Fig. 4.4.6. When only one Floquet multiplier is located on the unit circle in the complex plane, the periodic solution is known as **hyperbolic periodic solution**.

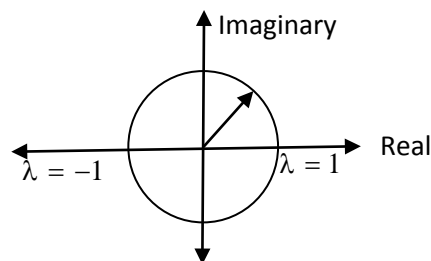


Fig. 4.4.4: Unit circle showing the real and imaginary parts of the eigenvalues.

If two or more Floquet multipliers are located on the unit circle in the complex plane then periodic response is known as non-hyperbolic periodic response. For $\lambda > 1$ the response is unstable. Similar to the bifurcation in the fixed point response, in periodic response also we have static and dynamic bifurcations. It should be noted that one of the Floquet multipliers associated with a periodic solution of an autonomous system of equations is always unity.

By changing the control parameter if a Floquet multiplier leaves the unit circle through $+1$, one may observe one of the following three bifurcations.

- Cyclic fold
- Symmetry breaking
- Transcritical Bifurcation

If Floquet multiplier leaves the unit circle through -1 period doubling bifurcation occurs.

Similarly if two complex conjugate Floquet multipliers leave the unit circle away from real axis, the resulting bifurcation is called secondary Hopf or Neimark bifurcation.

Cyclic fold or turning point bifurcation

Figure 4.4.7 shows the schematic diagram representing cyclic fold or turning point bifurcation in which the variation of amplitude of the periodic response is plotted with respect to the control parameter μ . Before the bifurcation point A, one observes a stable periodic and an unstable periodic response of the system which disappears after the bifurcation point. In some cases, a chaotic response may be observed after this cyclic fold bifurcation point and this behaviour of transition from periodic to chaotic response is termed as intermittent transition of type I to chaos. Hence, this type of bifurcations are dangerous, discontinuous and catastrophic type and the system should be operated below the critical bifurcation point.

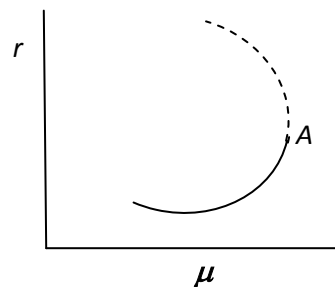


Fig. 4.4.7: Schematic diagram representing cyclic fold bifurcation.

Symmetry breaking bifurcation

Figure 4.4.8 shows the schematic diagrams in which the variations of amplitude of the periodic response with respect to the control parameter μ are plotted. In Fig. 4.4.8(a), before point A, there exists a single stable periodic response and after this point one obtains three periodic solutions out of which the original branch is unstable and other two branches are stable. These periodic solutions are

shown in phase portrait in Fig. 4.4.7 (a) and (b). When one obtains stable periodic solution before and after the bifurcation, the bifurcation is supercritical bifurcation which is depicted in Fig. 4.4.6 (a).

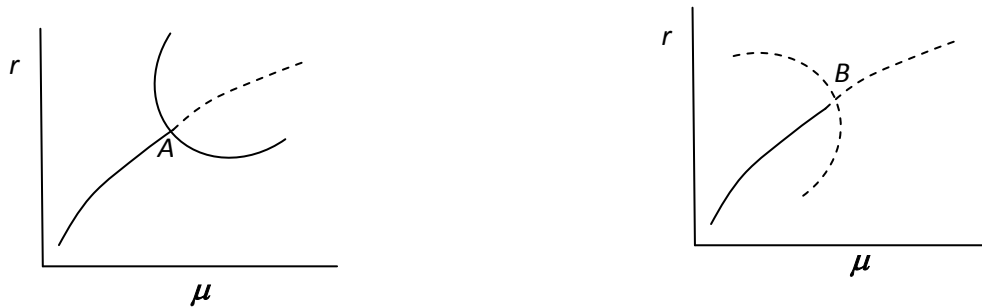


Fig. 4.4.8: (a) Supercritical symmetry breaking bifurcation, (b) subcritical symmetry breaking bifurcation

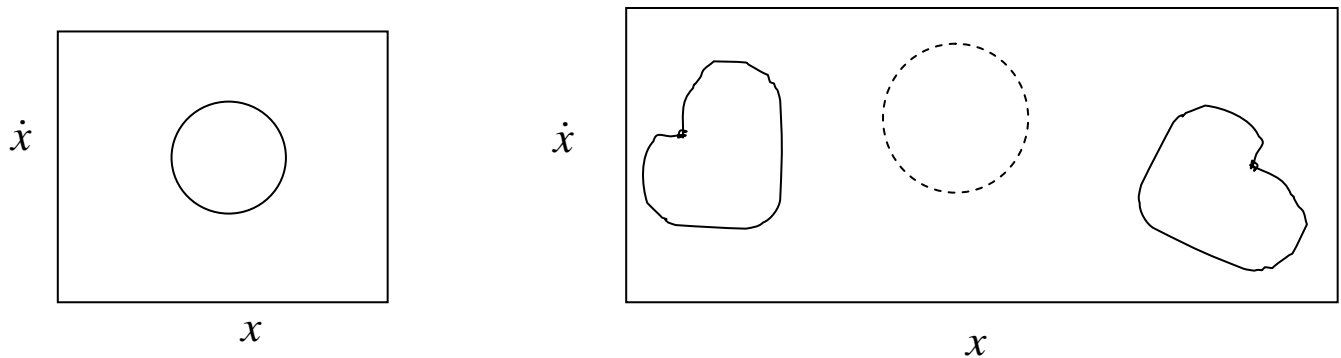


Fig. 4.4.8: (a) stable symmetric periodic response before bifurcation, (b) two stable unsymmetric and one symmetric unstable periodic response after bifurcation point.

A situation similar to that shown in Fig. 4.4.8 (b) may also occur where at bifurcation point B, a stable and two unstable periodic branches meet to give rise to an unstable periodic branch. As an unstable periodic response is not physically attainable, the system will have a tendency to jump to a remote attractor which may be at infinity giving rise to a catastrophic failure to the system. This bifurcation is known as subcritical symmetry breaking bifurcation.

Transcritical bifurcation

Similar to the discussion made in the bifurcation in the fixed point response, here also in transition bifurcation, the branches of stable and unstable periodic response exchange their stability as shown in Fig. 4.4.10.

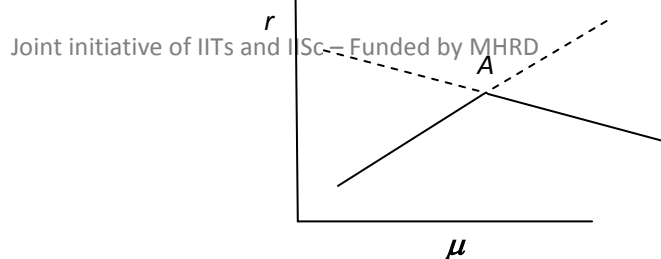


Fig. 4.4.10: Schematic diagram representing transcritical bifurcation

Period doubling or Flip bifurcation

In this case, the stable periodic solution branch that exists before the bifurcation point continues as an unstable branch and a new branch of solution having period doubled that of the original solution originates. If a stable branch originates, then the bifurcation is supercritical and if a branch of unstable period-doubled solutions is destroyed it is called subcritical bifurcation.

Secondary Hopf or Neimark Bifurcation

In this case, after the bifurcation the bifurcating solution may be periodic or two period quasi-periodic depending on the relation between the newly introduced frequency and the frequency of the original periodic solution that existed before the bifurcation. Here also one may have subcritical and supercritical bifurcations.

Exercise Problems:

Numerically solve the equations given in the following references for different system parameters

1. S. K. Dwivedy and R. C. Kar, Dynamics of a slender beam with an attached mass under combination parametric and internal resonances, Part II: Periodic and Chaotic response. *Journal of Sound and Vibration*, 222, no 2, 281-305, 1999.
2. S. K. Dwivedy and R. C. Kar, Dynamics of a slender beam with an attached mass under combination parametric and internal resonances, Part I: steady state response. *Journal of Sound and Vibration*, 221, no 5, 823-848, 1999

Module 4 Lecture 5

Quasi-periodic and Chaotic response

In the previous lectures in this module we have discussed about fixed point and periodic responses and in this lecture we will discuss about the quasi-periodic and chaotic responses.

Quasiperiodic Response

Consider the response $x(t)$ of a system which contains two frequency terms as given below.

$$x(t) = x_1 \sin \omega_1 t + x_2 \sin \omega_2 t$$

This response will be periodic if the ratio between the two frequencies is a rational number and if the frequencies are incommensurable i.e., if this ratio is an irrational number, then the response will be quasi-periodic. For example consider $\omega_1 = 2, \omega_2 = 2\sqrt{2}, x_1 = x_2 = 10$. In this case the ratio is $\sqrt{2}$ which an irrational number. The time response is shown in Figure 4.5.1.

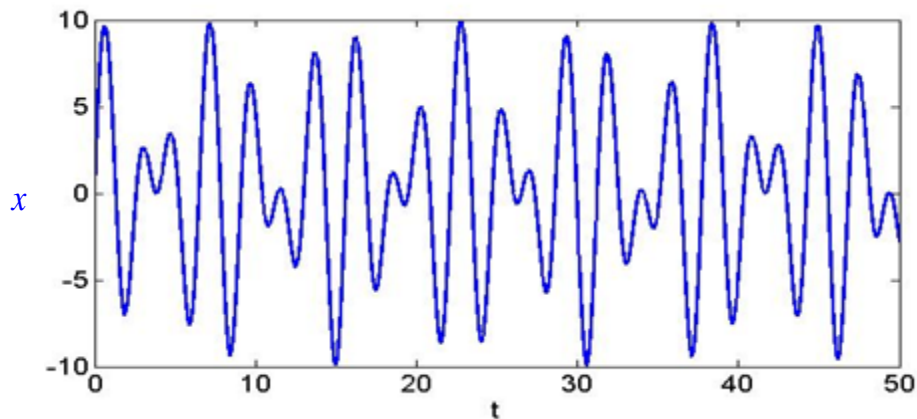


Fig. 4.5.1: Time response of a typical quasi-periodic response

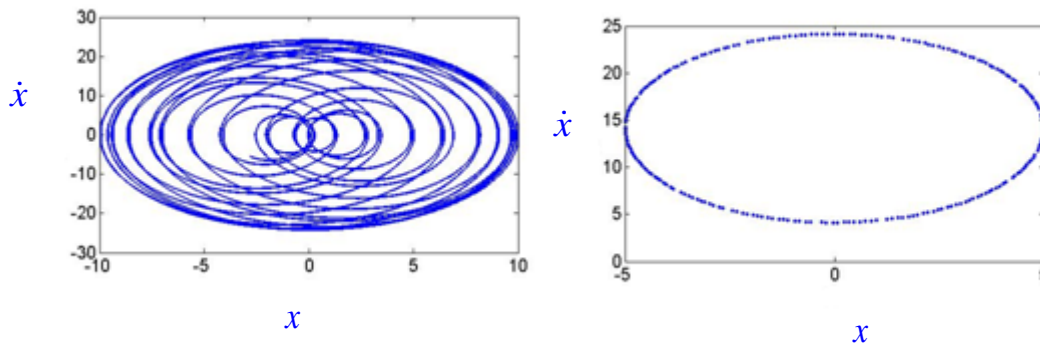


Fig. 4.5.2: (a) Phase portrait and Poincaré' section of the response

Few more examples are given in Figure 5.4.3 and 5.4.4 where the time response, phase portrait and the Poincaré' sections are plotted. It may be observed from the phase portrait that the structure is that of a torus and the Poincaré' section is a closed curve. It may be noted that in case of commensurable frequencies, one obtains multiple loops in the phase portraits and discrete points in the Poincaré section.

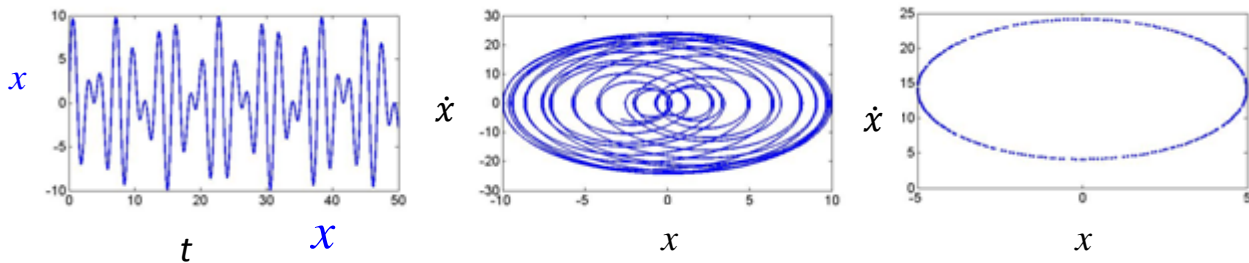


Fig. 4.5.3: (a) Time response, phase (b) portrait and (c) Poincaré' section for $x(t) = 5(\sin 2t + \sin 2\sqrt{2}t)$

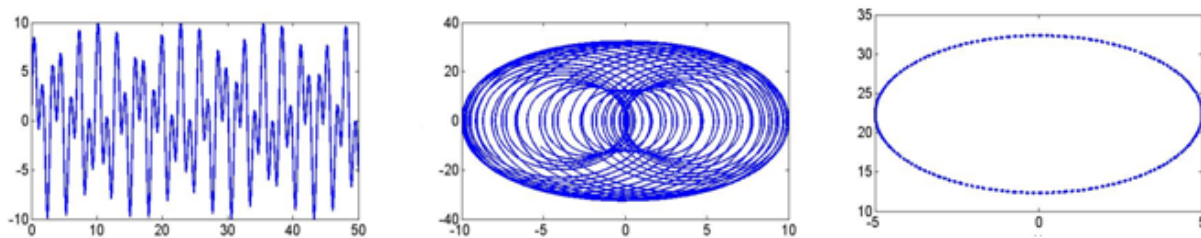


Fig. 4.5.4: (a) Time response, (b) phase portrait and (c) Poincaré' section for $x(t) = 10(\sin 2t + \sin 2\sqrt{5}t)$

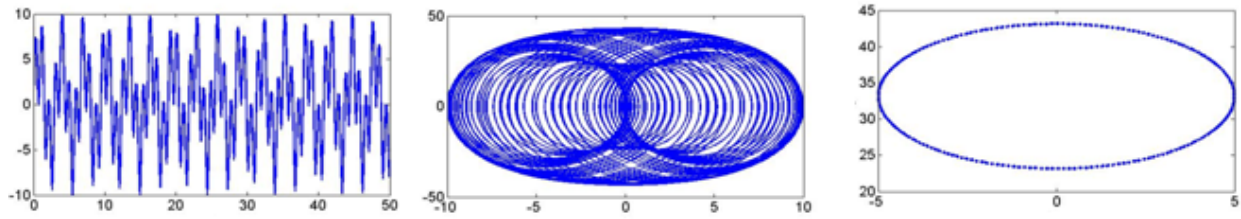


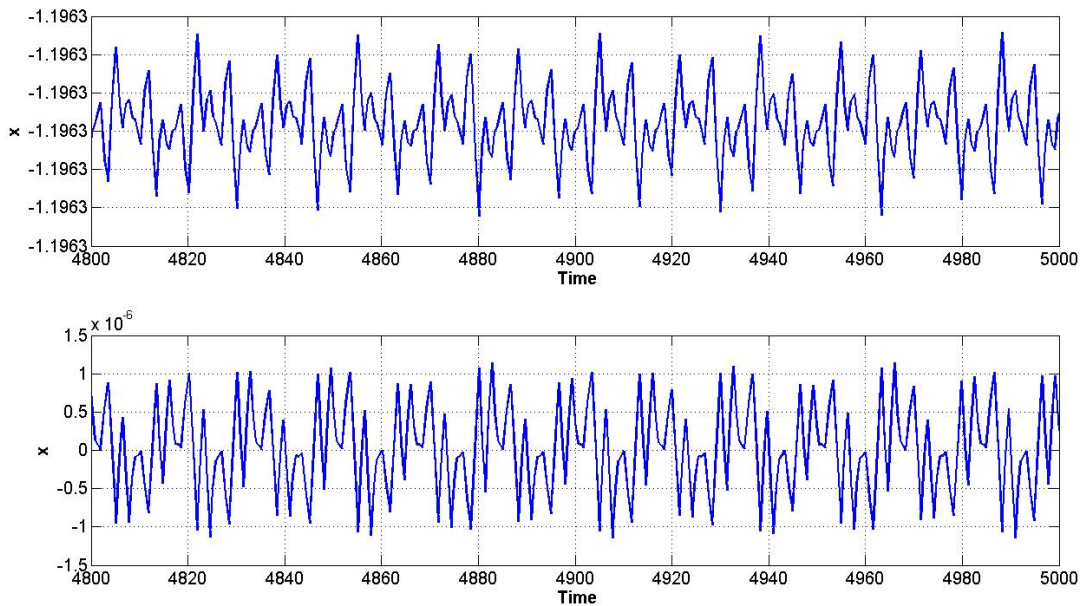
Fig. 4.5.4: (a) Time response, phase portrait and Poincaré section for $x(t) = 5(\sin 2t + \sin 2\sqrt{11}t)$

Similarly, one may solve the following Duffing equation with two forcing terms having incommensurable frequencies using Runge-Kutte method. One may use the ode45 function of Matlab for this purpose. The Matlab code is also given below and the obtained time response

$$\ddot{x} + x + 2\dot{x} + x^3 = \cos 2t + \cos 2\sqrt{2}t$$

$$\dot{x}_1 = x_2 \quad (x_1 = x)$$

$$\dot{x}_2 = -(x + 2\dot{x} + x^3) + \cos 2t + \cos 2\sqrt{2}t$$



```
clc
% Time Response and Phase portrait of Duffing Oscillator

global muglobal mu om1 om2 alpha
```

```

mu=0.8;
alpha=0.1
om1=2;
om2=2*sqrt(2);

[T,Y] = ode45(@ff22,[0 5000],[0.01 -0.3]);
ns=length(Y);
nm=floor(ns*0.9);
figure(1)
plot(Y(nm:ns,1),Y(nm:ns,2),'r','linewidth',2)
set(gca,'FontSize',15)
xlabel('\bf x','FontSize',15)
ylabel('\bf u','FontSize',15)
grid on

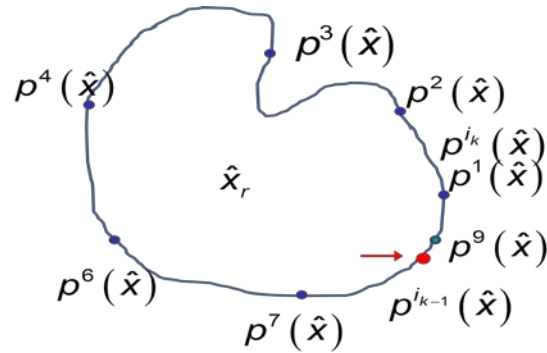
function dy = ff22(t,y)
global mu om1 om2 alpha

dy = zeros(2,1); % a column vector
dy(1) = y(2);
dy(2) =cos(om1)+cos(om2) -y(1)-alpha*y(1)^3-2*mu*y(2);
figure(2)
subplot(2,1,1)
plot(T(nm:ns),Y(nm:ns,1),'linewidth',2)
grid on
set(gca,'FontSize',15)
xlabel('\bf Time','FontSize',15)
ylabel('\bf x','FontSize',15)
subplot(2,1,2)

plot(T(nm:ns),Y(nm:ns,2),'linewidth',2)
grid on
set(gca,'FontSize',15)
xlabel('\bf Time','FontSize',15)
ylabel('\bf x','FontSize',15)

```

Rotational number



The discrete points on a Poincaré section of a two-period quasi-periodic orbit fall on a closed curve

\hat{x} initial point

$p^i(\hat{x})$ discrete point obtained

after i^{th} iteration of the Poincaré map P

Let i_{k-1}^{th} and i_k^{th} iterates bracket

\hat{x} after we go k times the close loop. Then

$$T_w = \lim_{k \rightarrow \infty} \frac{i_k}{k}$$

Winding number

Winding time represents the average number of iterates required to get back to \hat{x}

The inverse of the winding time is called the winding number or the rotational number.

$$\text{Rotational number } \rho = \frac{1}{T_w} = \frac{1}{2\pi} \lim_{k \rightarrow \infty} \sum_{i=1}^k \frac{\alpha_i}{k}$$

5.2 Chaotic Response

- A chaotic solution is a bounded steady-state behaviour that is not an equilibrium solution or periodic or quasi-periodic solution.
- Chaotic attractors are complicated geometrical objects that possess fractal dimensions.
- Unlike spectra of periodic and quasi-periodic attractors which consists of a number of sharp spikes, the spectrum of chaotic signal has a continuous broadband character.

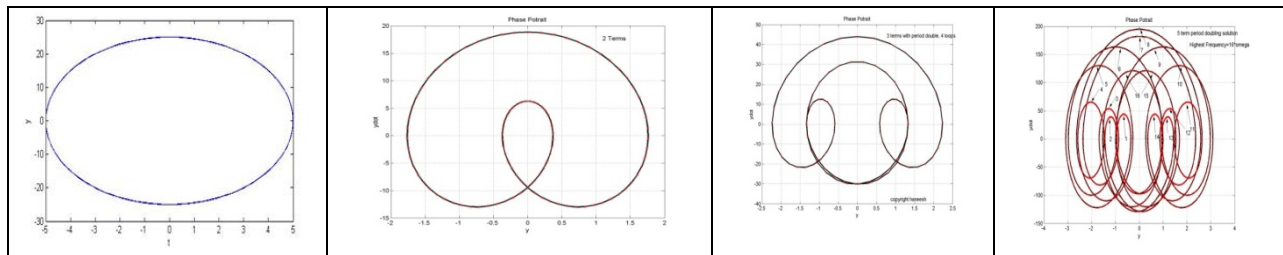
- In addition to the broadband components, the spectrum of a chaotic signal often contains spikes that indicate the predominant frequencies of the signal.
- Sensitive to initial condition: Butterfly effect
- A chaotic motion is the superposition of a very large number of unstable periodic motion.
- Thus a chaotic system may dwell for a brief time on a motion that is very nearly periodic and then may change to another periodic motion with period that is k times that of the preceding motion.
- This constant evolution from one periodic motion to another produces a long-time impression of randomness while showing a short-term glimpses of order.

Routes to Chaos

- Period doubling Route to Chaos
- Quasi-periodic route to chaos
- Intermittency and Crises route to chaos

Different Route to Chaos

5.2.1: Period doubling route to chaos

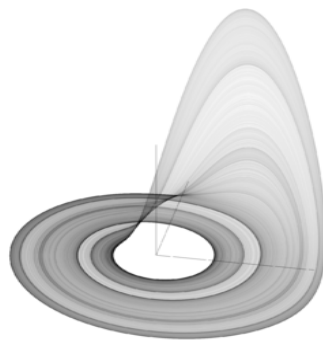
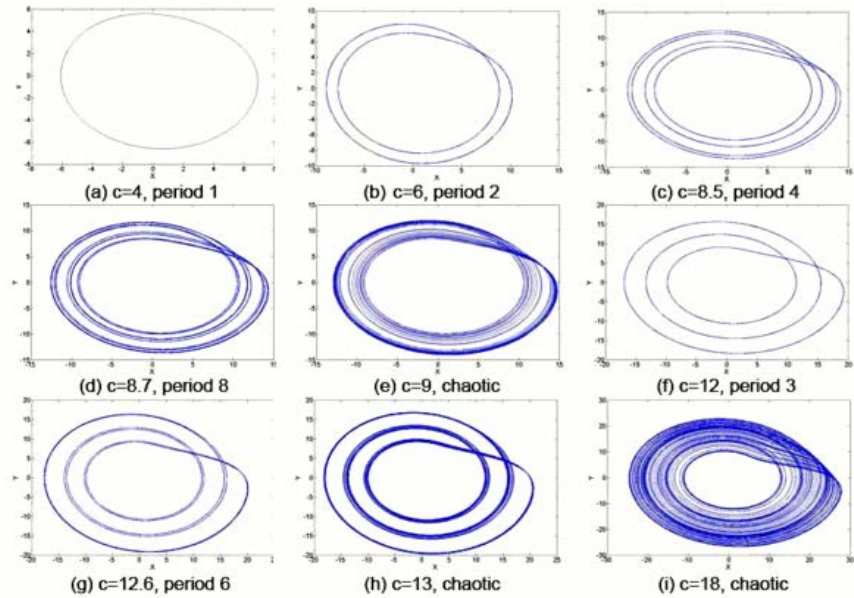


Rosler Equation

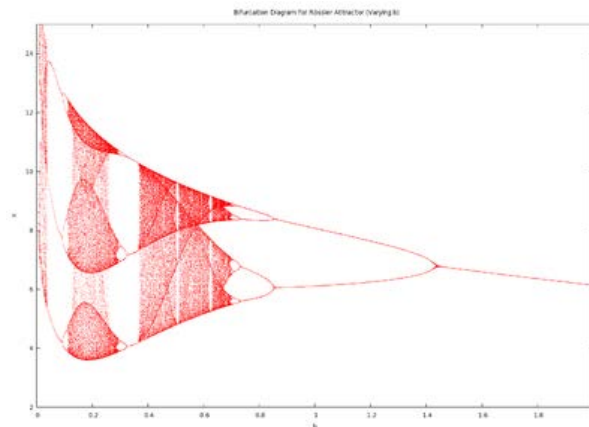
$$\dot{x} = -(y + z)$$

$$\dot{y} = x + ay$$

$$\dot{z} = b + (x - c)z$$



Wikipedia



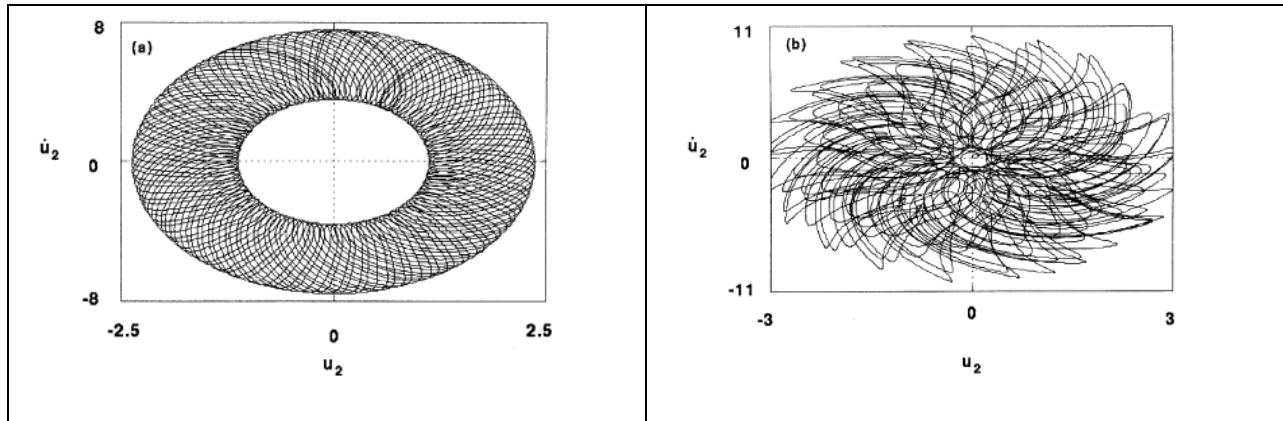
Feigenbaum number

Feigenbaum showed that the sequence of period doubling control parameter values scales according to the law

This number is same for all period-doubling sequence associated with smooth maps having a quadratic maximum

$$\delta = \lim_{k \rightarrow \infty} \frac{\alpha_k - \alpha_{k-1}}{\alpha_{k+1} - \alpha_k} = 4.66292016$$

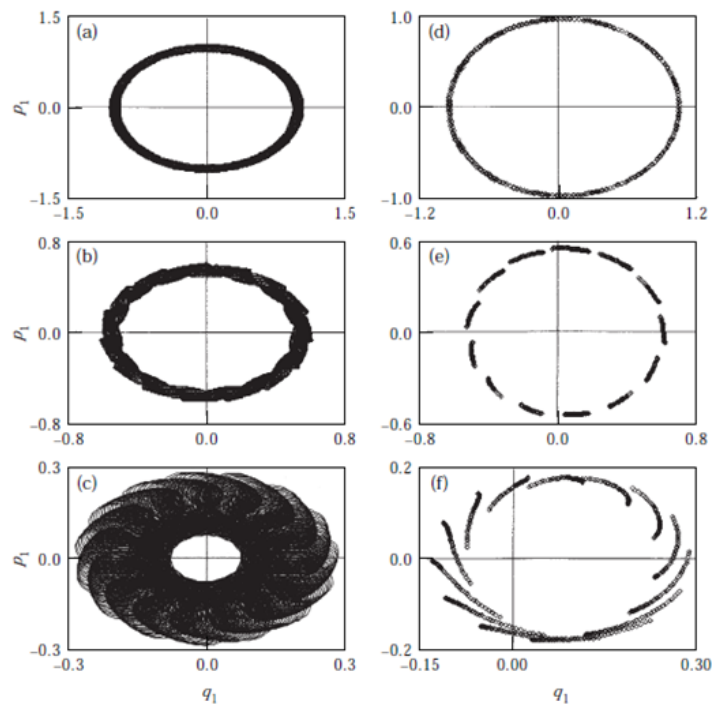
5.2.2: Quasi-periodic route to chaos (Torus doubling route to chaos)



Cascade of torus break down doubling route to chaos (a) Phase portrait of quasi-periodic orbit (b) chaotic orbit

$\phi = 2.13, \Gamma = 8$: (a) T_2 orbits at $\nu = 8.5$, (b) chaotic orbits at $\nu = 8.3$.

5.2.3 Torus breakdown route to chaos

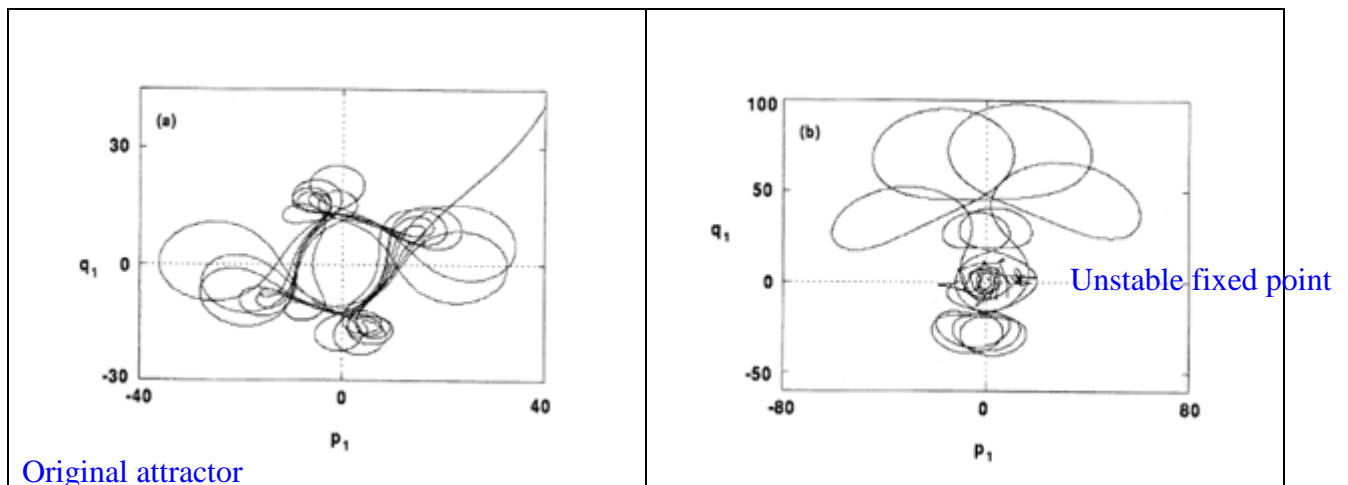


Different responses for $\phi = 3.5$, $\Gamma = 10$. (a) Quasi-periodic orbit with initial condition (ic) (1.0, -0.34, 3.14, 5.23); (b) chaotic response due to broken torus with ic (0.6, -0.34, 3.14, 6.28), and (c) chaotic response with ic (0.23, -0.34, 3.14, 6.28); (d), (e) and (f) are the corresponding Poincaré sections.

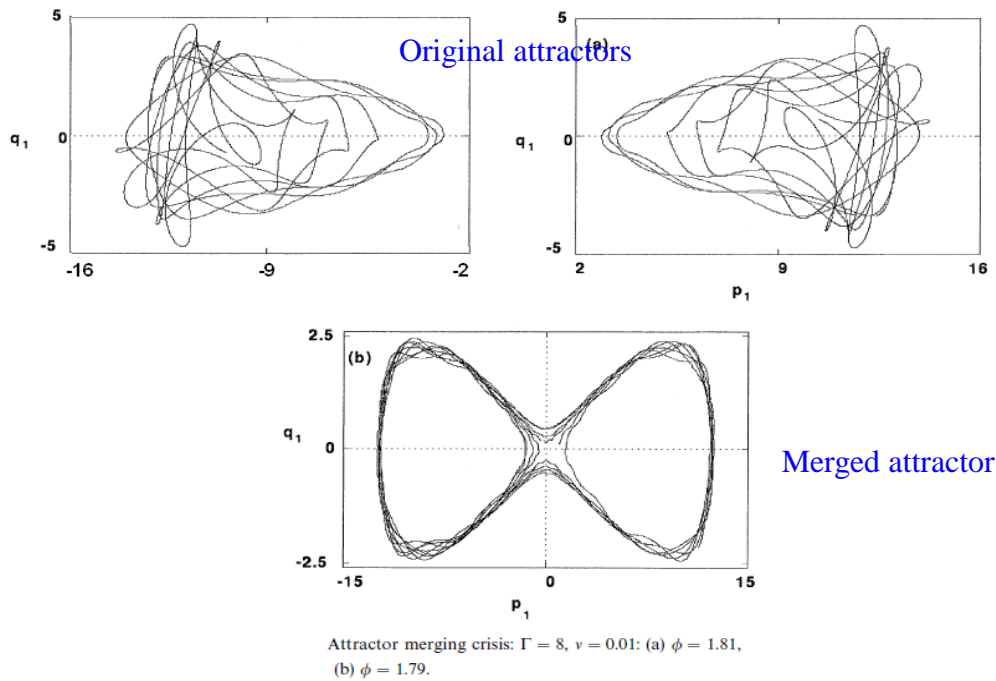
5.2.4 Crises

Interior Crises:

Chaotic attractor come in contact with unstable fixed point and explodes to form a larger chaotic attractor which contain the original attractor within it. Figure shows an interior crisis observed in a parametrically excited cantilever beam with arbitrary lumped with 1:3 internal resonance.



Interior crisis: Chaotic attractor for $\phi = 1.9$, $\Gamma = 8$: (a) $\nu = 0.5$, (b) $\nu = 1$. (Kar and Dwivedy 1999)



(Kar and Dwivedy 1999)

Attractor merging Crises: In this case two chaotic attractors come in contact to form a larger chaotic attractor giving rise to attractor merging crisis.

Exterior Crises This crisis occurs when a chaotic attractor suddenly destroyed as the control parameter passes through its critical value. The post bifurcation point may be a fixed-point, periodic, quasi-periodic or chaotic response. [Nayfeh and Balachandran, 1995]

References

1. F. Gao, H.L. Wang, Z.H. Wang, **Hopf bifurcation of a nonlinear delayed system of machine tool vibration via pseudo-oscillator analysis**, *Nonlinear Analysis: Real World Applications*, Volume 8, Issue 5, December 2007, Pages 1561-1568
2. R. C. Kar and S. K. Dwivedy, Non-linear dynamics of a slender beam carrying a lumped mass with principal parametric and internal resonances. *International Journal of Nonlinear Mechanics*, 34, (3), 515-529, 1999.

