ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-8 SIMILARITY SOLN TO TEMP BL - I

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1. Condition for Existence of Similarity Solutions
2. Similarity Equation and Boundary Conditions

BL Energy Equation L8($\frac{1}{11}$)

$$\rho C_p \left[u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \right] = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{dp_\infty}{d\,x} + \dot{Q}_{chem} + \dot{Q}_{rad} \quad (1)$$

Source Terms and Boundary Conditions:

1. $\dot{Q}_{chem}$ and $\dot{Q}_{rad}$ are presently neglected
2. $u\, dp_\infty / d\, x$ is important only in high-speed gas flows - presently neglected
3. at y = 0, T = T_w (x) (Wall Temperature)
4. as y $\rightarrow \infty$, T = T_∞ (Constant Free Stream Temperature)

Development of Similarity Eqn - L8($\frac{2}{11}$)

Define

$$T_w(x) - T_\infty = G(x) \quad (2)$$

$$\theta(\eta) = \frac{T(x,y) - T_\infty}{T_w(x) - T_\infty} \qquad \eta = y\sqrt{\frac{U_\infty}{\nu\, x}} \quad (3)$$

Then, the energy eqn will read as

$$\left[u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y}\right] + \frac{u\theta}{G}\frac{dG}{dx} = \alpha\frac{\partial^2\theta}{\partial y^2} + \frac{\nu}{Cp\,(T_w - T_\infty)}\left(\frac{\partial u}{\partial y}\right)^2 \quad (4)$$

Each term is now represented in similarity variables

Similarity Variables - L8($\frac{3}{11}$)

Recall the following definitions

$$
\begin{aligned}
U_\infty &= C\,x^m \\
\eta &= y\sqrt{\frac{U_\infty}{\nu\,x}} \qquad \psi = f\,(\eta)\,n(x) \qquad n(x) = \sqrt{\nu\,U_\infty\,x} \\
u &= U_\infty\,f' \qquad v = -\,\frac{\partial \psi}{\partial x} = -\left[f'\,n(x)\,\frac{\partial \eta}{\partial x} + f\,\frac{dn}{dx}\right] \\
\frac{\partial \theta}{\partial x} &= \theta'\,\frac{\partial \eta}{\partial x} \qquad \frac{\partial \theta}{\partial y} = \theta'\,\frac{\partial \eta}{\partial y} = \theta'\,\sqrt{\frac{U_\infty}{\nu\,x}} \\
\frac{\partial^2 \theta}{\partial y^2} &= \theta''\,\frac{U_\infty}{\nu\,x} \qquad (\frac{\partial u}{\partial y})^2 = \frac{U_\infty^3}{\nu\,x}\,(f'')^2
\end{aligned}
$$

Substitution gives (see next slide)

Similarity Equation - I - L8($\frac{4}{11}$)

$$f' \, \theta' \, \frac{\partial \eta}{\partial x} \quad - \quad \frac{\theta'}{n} \left[f' \, n \, \frac{\partial \eta}{\partial x} + f \, \frac{d\, n}{dx} \right] + \frac{f' \, \theta}{G} \frac{dG}{dx}$$

$$= \quad \frac{\theta''}{Pr\, x} + \frac{U_\infty^2}{Cp(T_w - T_\infty)} \, \frac{(f'')^2}{x}$$

or, upon simplification and multiplication by x

$$f' \, \theta \, (\frac{x}{G} \frac{dG}{dx}) - f \, \theta' \, (\frac{x}{n} \, \frac{dn}{dx}) = \frac{\theta''}{Pr} + 2 \, Ec_x \, (f'')^2$$

where Eckert Number $Ec_x = (U_\infty^2/2)/(Cp(T_w - T_\infty))$.

It can be shown that (x / n) (dn / dx) = (m + 1) /2.

Similarity Equation - II - L8($\frac{5}{11}$)

Hence the similarity equation will read as

$$\theta'' + Pr\left[(\frac{m+1}{2})\, f\, \theta' - f'\, \theta\, (\frac{x}{G}\frac{dG}{dx}) + 2\, Ec_x\, (f'')^2 \right] = 0 \qquad (5)$$

Similarity solutions are possible only when

1. (x/G)(dG/dx) = constant (γ, say) or
   G (x) = $T_w\,(x) - T_\infty$ = $\Delta\, T_{ref}\, x^\gamma$
2. $EC_x = (U_\infty^2(x)/2)/(Cp(T_w(x) - T_\infty))$ = constant or

$$EC_x = (\frac{C^2}{2Cp\,\Delta\, T_{ref}})\,(\frac{x^{2m}}{x^\gamma}) = constant$$

3. Hence, $\gamma = 2m$ when $Ec_x \neq 0$ (or, when viscous dissipation is accounted)

Final Similarity Equation - L8($\frac{6}{11}$)

Hence the final similarity equation will read as

$$\theta^{''} + Pr\,\left[(\frac{m+1}{2})\,f\,\theta^{'} - \gamma\,f^{'}\,\theta + 2\,Ec\,(f^{''})^2\right] = 0 \qquad (6)$$

where Ec = $(U_\infty^2(x)/2)/(Cp\,\Delta T_{ref}\,x^\gamma)$. If Ec $\neq 0$, $\gamma = 2m$

The Boundary Conditions are:

$$\theta(0) = 1 \quad \text{and} \quad \theta(\infty) = 0$$

Solution: $\theta(\eta) = F(m, B_f, Pr, \gamma, Ec)$ *If* $Ec \neq 0, \;\; \gamma = 2m$

Shooting Method - L8($\frac{7}{11}$)

The 2nd order equation is split into two 1st order ODEs

$$\frac{d\,\theta}{d\,\eta} = \theta^{'} \quad \text{with} \quad \theta\,(0) = 1 \quad \text{(known)} \tag{7}$$

$$\frac{d\,\theta^{'}}{d\,\eta} = \theta^{''} = -\,Pr\left[\left(\frac{m+1}{2}\right) f\,\theta^{'} - \gamma\, f^{'}\,\theta + 2\,Ec\,(f^{''})^2\right]$$
$$\text{with} \quad \theta^{'}\,(0) \quad (\text{ unknown }) \tag{8}$$

1. Solution of Velocity Boundary Layer gives $f, f^{'}, f^{''}$
2. Then, $\theta^{'}$ (0) is guessed and the two equations are solved by R-K method from $\eta = 0$ to $\eta = \eta_{max}$.
3. At each iteration, BC $\theta(\eta_{max}) \rightarrow 0$ is checked.
4. If NOT satisfied, $\theta^{'}$ (0) is revised

Output Parameters - I - L8($\frac{8}{11}$)

1. The *Physical Thickness* Δ is notionally associated with value of y where $\theta\,(\eta_{max}) \simeq 0.01$.
2. *Enthalpy Thickness* Δ_2 is defined as

$$\Delta_2 = \int_0^{\infty} \frac{\rho\, Cp\, u\,(T - T_\infty)}{\rho_\infty\, Cp_\infty\, U_\infty (T_w - T_\infty)}\, d\,y \tag{9}$$

3. Dimensionless Form (Uniform Property)

$$\Delta_2^* = \frac{\Delta_2}{x}\, Re_x^{0.5} = \int_0^{\eta_{max}} f'\,\theta\, d\,\eta \tag{10}$$

Output Parameters - II - L8($\frac{9}{11}$)

Local H T Coef, Nusselt and Stanton Numbers

$$h_x = \frac{q_w}{T_w - T_\infty} = -\frac{k\,(\partial T/\partial y)_{y=0}}{T_w - T_\infty} = -k\,\sqrt{\frac{U_\infty}{\nu\,x}}\,\theta'\,(0) \quad (11)$$

$$Nu_x = \frac{h_x\,x}{k} = -\,Re_x^{0.5}\,\theta'\,(0) \quad (12)$$

$$St_x = \frac{h_x}{\rho\,Cp\,U_\infty} = \frac{Nu_x}{Re_x\,Pr} \quad (13)$$

$$Nu_x, St_x = F(m, B_f, Pr, \gamma, Ec) \;\; If \;\; Ec \neq 0, \;\; \gamma = 2m$$

$$\overline{Nu} = \frac{\overline{h}\,x}{k} = (\frac{2}{m+1})\,Nu_L \quad \overline{h} = \frac{1}{L}\int_0^L h_x\,dx \quad (14)$$

Typical Profiles - L8($\frac{10}{11}$)

Flat Plate, No Suction/Blowing, Const Wall Temp, No Viscous Dissipation (Pr = 1)

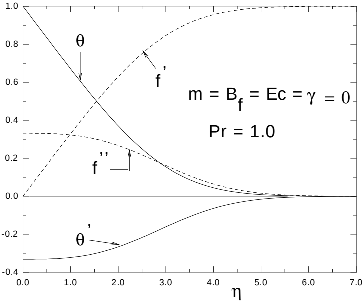

Figure: $f''' + 0.5\, f\, f'' = 0$ and $\theta'' + 0.5\, f\, \theta' = 0$.
$f(0) = f'(0) = 0$, $\theta(0) = 1$ and $f'(\infty) = 1$, $\theta(\infty) = 0$.
Hence, $\theta(\eta) = 1 - f'(\eta)$ - Perfect Analogy Between Heat and Momentum Transfer $\Delta^* = \delta^*$, $\Delta_2^* = \delta_2^*$ and $-\theta'(0) = f''(0)$

Moderate Pr Numbers - L8($\frac{11}{11}$)

$(m = B_f = \gamma = Ec = 0)$

Pr	0.7	1.0	5.0	10.0	25.0
$-\theta^{'}(0)$	0.291	0.330	0.572	0.721	0.976
Δ^*	5.60	4.92	2.73	2.15	1.59
Δ_2^*	0.834	0.663	0.231	0.146	0.0796

$-\theta^{'}(0) = Nu_x / Re_x^{0.5}$ can be correlated as

$$Nu_x = 0.332\, Re_x^{0.5}\, Pr^{0.33}$$

Very good agreement with Experimental data.

Δ^* decreases with increase in Pr. $\Delta^* = \delta^*$ for Pr = 1 .

In the next lecture, effects of other parameters are considered