ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-3 LAWS OF CONVECTION

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1. Fundamental Laws
2. Laws Governing Fluid Motion
3. Navier-Stokes Equations

Fundamental Laws - L3($\frac{1}{16}$)

1. Law of Conservation of Mass (Transport of Mass)
2. Newton's Second Law of Motion (Transport of Momentum)
3. First Law of Thermodynamics (Transport of Energy)

The first two laws define the fluid motion

The Laws are applied to an infinitesimally small *control-volume* located in a moving fluid.

Modeling a Fluid and its Motion - L3($\frac{2}{16}$)

There are Two approaches

1. PARTICLE APPROACH
2. CONTINUUM APPROACH

Particle Approach - L3($\frac{3}{16}$)

1. In the *Particle Approach* , the fluid is assumed to consist of particles (molecules, atoms) and the laws are applied to study particle motion. Fluid motion is then described by statistically averaged motion of a group of particles
2. For most applications arising in engineering and the environment, this approach is too cumbersome because the significant dimensions (L) of the flow (eg. Radius of a pipe or Boundary layer thickness) are considerably bigger than the Mean Free Path Length (MFL) between molecules.
3. The Avogadro's number specifies that at normal temperature (25 C) and pressure (1 atm), a gas will contain 6.022×10^{26} molecules per kmol. Thus in air, for example, there will be $\simeq 2 \times 10^{16}$ molecules per mm^3. MFL is very small indeed.

Continuum Approach - L3($\frac{4}{16}$)

1. In the *Continuum Approach* , therefore, statistical averaging is assumed to have been *already performed* and the fundamental laws are applied to portions of fluid (or, *control-volumes*) that contain a large number of particles.
2. The *information lost in averaging* must however be *recovered*.
3. This is done by invoking some further *auxiliary laws* and by empirical specifications of *transport properties*
   1. Viscosity μ , (Stokes's Stress-Strain Law)
   2. Thermal Conductivity k (Fourier's Law)
   3. Mass-Diffusivity D (Fick's Law)
4. The transport properties are typically determined from experiments.

Knudsen Number - L3($\frac{5}{16}$)

1. Knudsen Number Kn is defined as

$$Kn \equiv \frac{l}{L}$$

   where l is MFL and L is characteristic Flow-dimension
2. Continuum Approach is considered valid when $Kn < 10^{-4}$.
3. In Micro-Channels , Particle Approach becomes necessary because L is very small.

Control Volume Definition L3($\frac{6}{16}$)

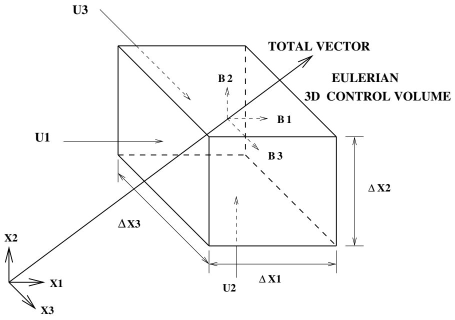
1. **Control Volume** (CV) is defined as
   A region in space across the boundaries of which matter, energy and momentum may flow and, it is a region *within* which source or sink of the same quantities may prevail. Further, it is a region on which external forces may act.
2. In general, a CV may be large or infinitesimally small. However, consistent with the idea of a *differential* in a continuum , an infinitesimally small CV is considered.
3. The CV is located *within* a moving fluid. Again, two approaches are possible:
   1. Lagrangian Approach
   2. Eulerian Approach

Lagrangian/Eulerian Approach L3($\frac{7}{16}$)

1. In the *Lagrangian Approach* , the CV is considered to be moving with the fluid as a whole.
2. In the *Eulerian Approach* , the CV is assumed *fixed* in space and the fluid is assumed to flow through and past the CV.
3. Except when dealing with certain types of unsteady flows (waves, for example), the Eulerian approach is generally used for its notional simplicity.
4. Measurements made using Stationary Instruments (Pitot Tube, Hot-wire, Laser-Doppler) can be directly compared with the solutions of differential equations obtained using the Eulerian approach.
5. We shall prefer Continuum + Eulerian Approach

Resolution of Total Vectors L3($\frac{8}{16}$)

1. The fundamental laws define *total flows of mass, momentum and energy* not only in terms of *magnitude* but also in terms of *direction* .
2. In a general problem of convection, neither magnitude nor direction are known *apriori* at different positions in the flowing fluid.



The problem of ignorance of direction is circumvented by resolving velocity, force and scalar fluxes in three directions that define the space.

Law of Mass Conservation -I L3($\frac{9}{16}$)

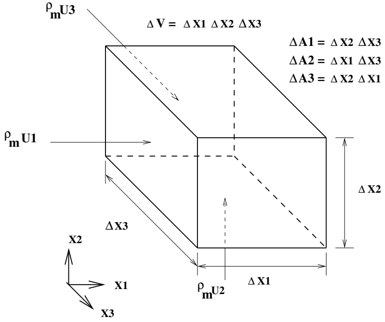
Statement

Rate of accumulation of mass ($\dot{M}_{ac}$) =
Rate of mass in ($\dot{M}_{in}$)
- Rate of mass out ($\dot{M}_{out}$)

$$\dot{M}_{ac} = \frac{\partial(\rho_m \, \Delta V)}{\partial t}$$

$$\dot{M}_{in} = \rho_m \, \Delta A_1 \, u_1 \, |_{x_1} + \rho_m \, \Delta A_2 \, u_2 \, |_{x_2} + \rho_m \, \Delta A_3 \, u_3 \, |_{x_3}$$

$$\dot{M}_{out} = \rho_m \, \Delta A_1 \, u_1 \, |_{x_1+\Delta x_1} + \rho_m \, \Delta A_2 \, u_2 \, |_{x_2+\Delta x_2} + \rho_m \, \Delta A_3 \, u_3 \, |_{x_3+\Delta x_3}$$



ρ_m = Bulk-Fluid or Mixture Density
Substitute and Divide each term by ΔV

Law of Mass Conservation -II L3($\frac{10}{16}$)

$$
\begin{aligned}
\frac{\partial \rho_m}{\partial t} &= \frac{\left(\rho_m\, u_1\,|_{x_1} - \rho_m\, u_1\,|_{x_1+\Delta x_1}\right)}{\Delta x_1} + \frac{\left(\rho_m\, u_2\,|_{x_2} - \rho_m\, u_2\,|_{x_2 \Delta x_2}\right)}{\Delta x_2} \\
&+ \frac{\left(\rho_m\, u_3\,|_{x_3} - \rho_m\, u_3\,|_{x_3+\Delta x_3}\right)}{\Delta x_3}
\end{aligned}
$$

Let $\Delta x_1, \Delta x_2, \Delta x_3 \rightarrow 0$

$$
\frac{\partial \rho_m}{\partial t} + \frac{\partial(\rho_m\, u_1)}{\partial x_1} + \frac{\partial(\rho_m\, u_2)}{\partial x_2} + \frac{\partial(\rho_m\, u_3)}{\partial x_3} = 0 \qquad (1)
$$

Alternate Non-Conservative Form

$$
\begin{aligned}
\frac{\partial \rho_m}{\partial t} + u_1 \frac{\partial \rho_m}{\partial x_1} + u_2 \frac{\partial \rho_m}{\partial x_2} + u_3 \frac{\partial \rho_m}{\partial x_3} &= -\rho_m \left[\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right] \\
\frac{D\, \rho_m}{D\, t} &= -\rho_m\, \bigtriangledown\, .\, V \qquad (2)
\end{aligned}
$$

Newton's Second Law of Motion - I L3($\frac{11}{16}$)

Statement

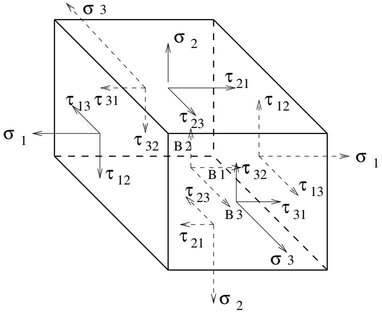
For a **Given Direction**

Rate of accumulation of momentum ($M\dot{o}m_{ac}$) =

Rate of momentum in ($M\dot{o}m_{in}$)

- Rate of momentum out ($\dot{M}_{out}$)

+ Sum of forces acting on the CV (F_{cv})



τ - shear stresses (N / m^2)

σ - normal stresses (N / m^2)

B - Body forces (N / kg)

3 equations in 3 directions

Newton's Second Law of Motion - II L3($\frac{12}{16}$)

In Direction-1

$$
\begin{aligned}
Mom_{ac} &= \frac{\partial(\rho_m\,\Delta V\,u_1)}{\partial t} \\
Mom_{in} &= (\rho_m\,\Delta A_1\,u_1)\,u_1\,|_{x_1} + (\rho_m\,\Delta A_2\,u_2)\,u_1\,|_{x_2} \\
&+ (\rho_m\,\Delta A_3\,u_3)\,u_1\,|_{x_3} \\
Mom_{out} &= (\rho_m\,\Delta A_1\,u_1)\,u_1\,|_{x_1+\Delta x_1} + (\rho_m\,\Delta A_2\,u_2)\,u_1\,|_{x_2+\Delta x_2} \\
&+ (\rho_m\,\Delta A_3\,u_3)\,u_1\,|_{x_3+\Delta x_3} \\
F_{cv} &= -\,(\sigma_1\,|_{x_1} - \sigma_1\,|_{x_1+\Delta x_1}\,)\,\Delta A_1 \\
&+ (\tau_{21}\,|_{x_2+\Delta x_2} - \tau_{21}\,|_{x_2}\,)\,\Delta A_2 \\
&+ (\tau_{31}\,|_{x_3+\Delta x_3} - \tau_{31}\,|_{x_3}\,)\,\Delta A_3 \\
&+ \rho_m\,B_1\,\Delta V
\end{aligned}
$$

Newton's Second Law of Motion - III L3($\frac{13}{16}$)

In Direction-1 Substitute, Divide each term by ΔV and let $\Delta x_1, \Delta x_2, \Delta x_3 \rightarrow 0$

$$\frac{\partial(\rho_m\, u_1)}{\partial t} + \frac{\partial(\rho_m\, u_1\, u_1)}{\partial x_1} + \frac{\partial(\rho_m\, u_2\, u_1)}{\partial x_2} + \frac{\partial(\rho_m\, u_3\, u_1)}{\partial x_3} = \frac{\partial(\sigma_1)}{\partial x_1} + \frac{\partial(\tau_{21})}{\partial x_2} + \frac{\partial(\tau_{31})}{\partial x_3} + \rho_m\, B_1 \qquad (3)$$

This is Momentum equation in X_1 direction
LHS $\equiv$ Net Rate of Change of Momentum in X_1 direction
RHS $\equiv$ Net Forces in X_1 direction

Exercise: Similar procedure in Directions 2 and 3.

Tensor Notation L3($\frac{14}{16}$)

Mass Conservation equation

$$\frac{\partial(\rho_m)}{\partial t} + \frac{\partial(\rho_m\, u_j)}{\partial x_j} = 0 \tag{4}$$

Momentum equation in X_i direction (3 equations)

$$\frac{\partial(\rho_m\, u_i)}{\partial t} + \frac{\partial(\rho_m\, u_j\, u_i)}{\partial x_j} = \frac{\partial}{\partial x_i}\, [\sigma_i \delta_{ij}] + \frac{\partial}{\partial x_j}\, [\tau_{ji}\, (1 - \delta_{ij})] + \rho_m\, B_i \tag{5}$$

for i = 1,2,3 and j = 1,2,3 (cyclic). δ_{ij} = kronecker delta

Closure Problem: 4 equations and 12 unknowns

u_i (3), σ_i (3), τ_{ij} (6)

Stokes's Stress-Strain Laws L3($\frac{15}{16}$)

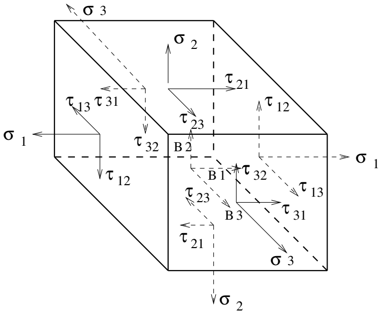
1. Shear Stress

$$\tau_{ij} = \mu \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] \quad (6)$$

2. Therefore, $\tau_{ij} = \tau_{ji}$
(Complementary Stress)
3. Normal Stress (Tensile)

$$\sigma_i = -p + 2\,\mu \frac{\partial u_i}{\partial x_i} \quad (7)$$

$$= -p + \tau_{ii} \quad (8)$$



Now, we have 4 equations and 4 unknowns: u_i (3) and p. Fluid Viscosity μ must be supplied. See next slide

Navier - Stokes Equations L3($\frac{16}{16}$)

Mass Conservation equation

$$\frac{\partial(\rho_m)}{\partial t} + \frac{\partial(\rho_m\, u_j)}{\partial x_j} = 0 \tag{9}$$

Momentum equation in X_i direction (3 equations)

$$\begin{aligned} \frac{\partial(\rho_m\, u_i)}{\partial t} + \frac{\partial(\rho_m\, u_j\, u_i)}{\partial x_j} &= -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\, \frac{\partial u_i}{\partial x_j}\right] \\ &+ \rho_m\, B_i + \frac{\partial}{\partial x_j}\left[\mu\, \frac{\partial u_j}{\partial x_i}\right] \end{aligned} \tag{10}$$

These are known as Navier - Stokes Equations . They describe fluid motion completely.