ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-26 TURBULENCE MODELS-1

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1. Main Task
2. Eddy Viscosity Models
   1. General Mixing Length model
   2. High Re_t One-Eqn model
   3. High Re_t Two-Eqn model

Main Tasks - L26($\frac{1}{20}$)

1. In multidimensional turbulent wall flows, even if inner-layer universality is exploited , RANS eqns must be solved in the outer layers through modeling
2. Thus, turbulent stresses $\overline{u_i' u_j'}$ and heat fluxes $\overline{\rho\, cp\, u_i'\, T'}$ must be modeled to recover lost information through averaging
3. This recovery must be carried out in a general way so that the model need not be changed from one flow situation to another
4. Imparting absolute generality to turbulence models has, however, turned out to be quite a difficult task.
5. Thirdly, the model must be economical; that is, the computational expense (and ease) must not be very much in excess of that which would be required for computation of a laminar flow under the same situation.

Two Main Approaches - L26($\frac{2}{20}$)

1. Relating $\overline{u_i' u_j'}$ to the mean rate of strain S_{ij} through a property called the *turbulent*- or *eddy*- viscosity μ_t. This approach derives its inspiration from the Stokes's stress-strain relations.
2. Recovering distribution of stresses from solution of transport equations for $\overline{u_i' u_j'}$ in which convection and diffusion of this quantity is principally balanced by rates of its production and dissipation.

In this lecture, we shall consider the most popular Eddy-viscosity models

The main idea - L26($\frac{3}{20}$)

1. The notion of μ_t introduced by Boussinesque can be generalised to read as:

$$-\rho\,\overline{u_i'\,u_j'} = \mu_t\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] - \frac{2}{3}\,\rho\,e\,\delta_{ij}$$

   where, μ_t is a property of the flow; not that of a fluid . μ_t is isotropic although its magnitude may vary with the position in the flow .

2. The term involving Kronecker delta δ_{ij} simply ensures that the sum of the normal stresses (i = j) will equal 2 ρ e and thus, the definition of TKE is retrieved since the sum of the strain rates ($\partial u_i/\partial x_i = 0$) is zero from requirement of continuity.

3. Using this model, number of unknowns (u_i and p) equals number of RANS eqns when μ_t is modeled.

Characterising μ_t - L26($\frac{4}{20}$)

1. From kinetic theory, laminar viscosity μ is written in dimensionally correct form as

$$\mu = \rho \times l_{\text{mfpl}} \times \overline{u}_{mol}$$

where l_{mfpl} is mean free path length and $\overline{u}_{mol}$ is average molecular velocity.

2. Analogously, μ_t is modeled as

$$\mu_t = \rho \times L \times |u^{'}|$$

where L and $|u^{'}|$ are representative length and velocity scales of turbulent fluctuations.

3. The main task now is to represent L and $|u^{'}|$ universally.

General Mixing Length Model - L26($\frac{5}{20}$)

1. Simplest model - $|u'| \propto l_m \times$ |mean velocity gradient|
2. Hence,

$$\mu_t = \rho_m \, l_m^2 \left\{ \frac{\partial u_i}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right\}^{0.5} \quad \text{summation,}$$

where $L = l_m$ is called *Prandtl's mixing length*

$$\begin{aligned} l_m &= \kappa \times n \times (1 - e^{-\xi}) \quad \xi = \frac{1}{26} \frac{n}{\nu} \sqrt{\frac{\tau_w}{\rho_m}} = \frac{n^+}{26} \\ l_m &= 0.2 \times \kappa \times R \, , \quad \text{if } \; n^+ > 26 \, , \end{aligned}$$

where n is the *normal distance* from the nearest wall and $\kappa \simeq 0.41$, R is a characteristic dimension, and wall shear stress τ_w is evaluated from the product of μ and total vel gradient $\partial V / \partial n|_{\text{wall}}$

l_m for 2D Boundary Layers - L26($\frac{6}{20}$)

1. For 2D BLs, $\mu_t = \rho_m \, l_m^2 \, (\partial u / \partial y)$
2. For inner and outer wall boundary layers, from lecture 24

$$l_m = \kappa \times y \times (1 - e^{-\frac{y^+}{A^+}})$$
$$A^+ = \frac{25}{a\left[v_w^+ + b\left\{\frac{p^+}{1+c\,v_w^+}\right\}\right] + 1}$$

3. For Free Shear Layers such as jets and wakes,

$$l_m = \beta \, y_{1/2}$$

where $y_{1/2}$ is the half-width of the shear layer and β = 0.225 (a plane jet), = 0.1875 (a round jet), = 0.40 (a plane wake).

One Eqn Model - 1 - L26($\frac{7}{20}$)

1. The mixing length model predicts that $-\rho \overline{u_i' u_j'} = 0$ where the strain rate $S_{ij} = 0$. In many situations this is not found. For example, in an annulus with outer wall rough and inner wall smooth, the plane of zero shear stress is closer to the smooth wall than the plane of zero vel gr.
2. Hence, we take the fluctuating velocity scale $|u'| = \sqrt{e}$. Then, $\mu_t = \rho \times L \times \sqrt{e}$ where L = Integral Length Scale .
3. The distribution of TKE (e) is determined from

$$\rho \left[\frac{\partial e}{\partial t} + u_j \frac{\partial e}{\partial x_j} \right] = -\frac{\partial}{\partial x_j} \left[\overline{u_j' (p' + \rho \frac{u_i' u_i'}{2})} - \mu \frac{\partial e}{\partial x_j} \right] + (-\rho \overline{u_i' u_j'}) \frac{\partial u_i}{\partial x_j} - \mu \overline{(\frac{\partial u_i'}{\partial x_j})^2}$$

Modeling TKE - L26($\frac{8}{20}$)

1. The turbulent diffusion term cannot be directly measured because no probe can simultaneously measure pressure and velocity fluctuations
2. But, noting the redistributive character of this term, we may assume gradient diffusion. Hence,

$$-\overline{u_j'(p' + \rho\,\frac{u_i'u_i'}{2})} = \frac{\mu_t}{\sigma_e}\,\frac{\partial e}{\partial x_j}$$

   where, σ_e is a turbulent Prandtl number for TKE.
3. Recall that when Re_t is high, dissipation can be represented in terms of large scale fluctuations. Hence,

$$\mu\,\overline{(\frac{\partial u_i'}{\partial x_j})^2} = C_D\,\frac{\rho\,e^{3/2}}{L} = \rho\,\epsilon$$

Modeled TKE Eqn L26($\frac{9}{20}$)

Replacing $-\rho\,\overline{u_i'\,u_j'} = \mu_t\,S_{ij}$

$$\rho\left[\frac{\partial e}{\partial t} + u_j\frac{\partial e}{\partial x_j}\right] = \frac{\partial}{\partial x_j}\left[(\mu + \frac{\mu_t}{\sigma_e})\,\frac{\partial e}{\partial x_j}\right] + \mu_t\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]\frac{\partial u_i}{\partial x_j} - C_D\,\frac{\rho\,e^{3/2}}{L}$$

where C_D and σ_e are expected to be *universal constants* when $Re_t = L\sqrt{e}/\nu$ is high (that is away from the wall and beyond the transition layer).

Determination of C_D and σ_e - L26($\frac{10}{20}$)

1. Recall that in the fully turbulent inner layer, production $\simeq$ dissipation (lecture 24) and equilibrium conditions prevail.
2. In this layer, $\tau_t = \mu_t \, \partial V / \partial n$. Hence

$$\tau_t \, \frac{\partial V}{\partial n} = \rho \, \sqrt{e} \, L \, (\frac{\partial V}{\partial n})^2 = C_D \, \frac{\rho \, e^{3/2}}{L}$$

   where V is vel. parallel to the wall and n is normal distance.
3. Also, in this layer $\tau_t \simeq \tau_w$. Hence, $(\tau_w / \rho) / \, e = u_\tau^2 / \, e = \sqrt{C_D}$
4. Recall that $\tau_w \simeq 0.3 \, \rho \, e$. Hence, $C_D \simeq 0.09$
5. σ_e is taken as 1.0 from numerical experiments in several flow situations.
6. Hence, the modeled TKE eqn can be solved. We must now specify L.

Specification of L - L26($\frac{11}{20}$)

1. L is determined as follows. Consider

$$\mu_t \, (\frac{\partial V}{\partial n})^2 = C_D \, \frac{\rho \, e^{3/2}}{L} \quad \text{(equilibrium condition)}$$
$$\mu_t = \rho \, e^{0.5} \, L \quad \text{(definition)}$$

2. Eliminating e from these two equations,
   $\mu_t = C_D^{-0.5} \, \rho \, L^2 \, (\partial V / \partial n)$
3. Comparing this equation with eqn with mixing length model

$$L = C_D^{0.25} \, l_m = 0.5477 \, l_m \quad \rightarrow \quad l_m = \kappa \, y$$

4. With these specifications of L, μ_t, C_D and σ_e, TKE equation can be solved along with the RANS momentum equations. In general flows, however, further refinements are needed.

Two-Equation Model - L26($\frac{12}{20}$)

1. For the more general case of turbulent flows involving strong and variable pressure gradients, flow recirculation, effects of swirl or buoyancy etc.,it is necessary to devise means for determining distribution of L in multidimensional flows.
2. Dividing the energy spectrum e(k) by wave number k and integrating from 0 to ∞, an equation for L in the spectral space can indeed be derived since, $L = \frac{1}{e} \int_0^\infty \frac{e(k)}{k} \, dk$. However, the eqn is not tractable in the physical space.
3. A length-scale equation, however, need not necessarily have L itself as its dependent variable; any combination of the form $Z = e^m L^n$ will suffice since e can be known from solution of modeled TKE equation

Proposals for $Z = e^m L^n$ Eqn - L26($\frac{13}{20}$)

$$\rho \left[\frac{\partial Z}{\partial t} + u_j \frac{\partial Z}{\partial x_j} \right] = \frac{\partial}{\partial x_j} \left[(\mu + \frac{\mu_t}{\sigma_Z}) \frac{\partial Z}{\partial x_j} \right] + C_1 \frac{Z}{e} P - C_2 Z \frac{\rho \sqrt{e}}{L} + S\,(Z)$$

where, $P = -\rho \overline{u_i' u_j'}\, \partial u_i / \partial x_j$ and C_1, C_2 are constants when $Re_t = e^{0.5} L/\nu$ is high.

Proposals for $Z = e^m L^n$		
m	n	Remark
3/2	-1	Dissipation Rate (ϵ Eqn)
1	1	(eL Eqn)
1	- 2	Vorticity Fluctuation (e / L^2 Eqn)
1/2	1	Turbulent viscosity ($e^{0.5}$ L Eqn)

Computational experience suggests that for $Z = (e^{3/2} / L) \propto \epsilon$, S (Z) = 0 and σ_Z = const . Hence, Dissipation eqn preferred .

Dissipation eqn - 1 - L26($\frac{14}{20}$)

At high Re_t, local isotropy prevails. Hence an eqn for $\epsilon = \nu\, \overline{(\partial u_i'/\partial x_j)^2}$ can be derived by differentiating N-S Eqn for u_i' w.r.t. x_j and then multiplying by $2\,\nu\,(\partial u_i'/\partial x_j)$. Time averaging gives exact ϵ Eqn .

$$
\begin{aligned}
\left[\frac{\partial \epsilon}{\partial t} + u_k \frac{\partial \epsilon}{\partial x_k}\right] &= -\nu\,\frac{\partial}{\partial x_k}\left[\overline{u_k'\,(\frac{\partial u_i'}{\partial x_j})^2} + \frac{1}{\rho}\overline{\frac{\partial p'}{\partial x_j}\,\frac{\partial u_k'}{\partial x_j}} - \frac{\partial \epsilon}{\partial x_k}\right] \\
&- 2\,\nu\left[\overline{\frac{\partial u_i'}{\partial x_j}\,\frac{\partial u_k'}{\partial x_j}} + \overline{\frac{\partial u_j'}{\partial x_i}\,\frac{\partial u_j'}{\partial x_k}}\right]\frac{\partial u_i}{\partial x_k} \\
&- 2\,\nu\,\overline{\frac{\partial u_i'}{\partial x_k}\,\frac{\partial u_i'}{\partial x_j}\,\frac{\partial u_k'}{\partial x_j}} - 2\left[\nu\,\overline{\frac{\partial^2 u_i'}{\partial x_k\,\partial x_j}}\right]^2
\end{aligned}
$$

Dissipation eqn - 2 - L26($\frac{15}{20}$)

Complex correlations can only be discerned from DNS.

$$\text{Diffusion} = -\nu \frac{\partial}{\partial x_k} \overline{\left(u_k' \left(\frac{\partial u_i'}{\partial x_j}\right)^2\right.} + \overline{\left.\frac{1}{\rho}\frac{\partial p'}{\partial x_j}\frac{\partial u_k'}{\partial x_j}\right)}$$

$$= \frac{\partial}{\partial x_k}\left\{C_3 \frac{e^2}{\epsilon}\frac{\partial \epsilon}{\partial x_k}\right\} \rightarrow \overline{u_j' u_k'} \propto e \text{ (assumption)}$$

$$\text{Generation} = -2\,\nu \left(\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_k'}{\partial x_j}} + \overline{\frac{\partial u_j'}{\partial x_i}\frac{\partial u_j'}{\partial x_k}}\right)\frac{\partial u_i}{\partial x_k}$$

$$= C_1\, \overline{u_i' u_j'}\, \frac{\epsilon}{e}\frac{\partial u_i}{\partial x_j}$$

$$\text{last 2 terms} = -2\,\nu\, \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_j}\frac{\partial u_k'}{\partial x_j}} - 2\,\overline{\left(\nu \frac{\partial^2 u_i'}{\partial x_k\, \partial x_j}\right)^2}$$

$$= C_2 \frac{\epsilon^2}{e} \text{ (Relevant in inertial subrange)}$$

Dissipation eqn - 3 - L26($\frac{16}{20}$)

Modeled dissipation equation

$$\left[\frac{\partial \epsilon}{\partial t} + u_k \frac{\partial \epsilon}{\partial x_k}\right] = \frac{\partial}{\partial x_k}\left\{(\nu + C_3 \frac{e^2}{\epsilon}) \frac{\partial \epsilon}{\partial x_k}\right\} - C_1 \overline{u_i' u_k'} \frac{\epsilon}{e} \frac{\partial u_i}{\partial x_k} - C_2 \frac{\epsilon^2}{e}$$

$$\text{But} \quad \mu_t = \rho \sqrt{e}\, L = C_D \frac{\rho\, e^2}{\epsilon} \quad \text{Hence,}$$

$$\rho \left[\frac{\partial \epsilon}{\partial t} + u_k \frac{\partial \epsilon}{\partial x_k}\right] = \frac{\partial}{\partial x_k}\left\{(\mu + \frac{\mu_t}{\sigma_\epsilon}) \frac{\partial \epsilon}{\partial x_k}\right\} + C_1 (-\rho\, \overline{u_i' u_k'}) (\frac{\epsilon}{e}) \frac{\partial u_i}{\partial x_k} - C_2 \frac{\rho\, \epsilon^2}{e}$$

where $-\rho\, \overline{u_i' u_k'} = \mu_t\ [\partial u_i/\partial x_k + \partial u_k/\partial x_i]$ and $\sigma_\epsilon \equiv$ turbulent Prandtl number for dissipation rate. It absorbs constants C_D and C_3.

High Re_t e-ϵ Model Eqns - L26($\frac{17}{20}$)

$$\rho \frac{D\,e}{Dt} = \frac{\partial}{\partial x_j}\left[(\mu + \frac{\mu_t}{\sigma_e})\frac{\partial e}{\partial x_j}\right] + \mu_t \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]\frac{\partial u_i}{\partial x_j} - \rho\,\epsilon$$

$$\rho \frac{D\,\epsilon}{Dt} = \frac{\partial}{\partial x_j}\left[(\mu + \frac{\mu_t}{\sigma_\epsilon})\frac{\partial \epsilon}{\partial x_j}\right] + \frac{\epsilon}{e}\left\{C_1\,\mu_t\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]\frac{\partial u_i}{\partial x_j} - C_2\,\rho\,\epsilon\right\}$$

Boundary layer Forms

$$\rho\left[\frac{\partial e}{\partial t} + u\,\frac{\partial e}{\partial x} + v\,\frac{\partial e}{\partial y}\right] = \frac{\partial}{\partial y}\left\{(\mu + \frac{\mu_t}{\sigma_e})\frac{\partial e}{\partial y}\right\} + \mu_t\,(\frac{\partial u}{\partial y})^2 - \rho\,\epsilon$$

$$\rho\left[\frac{\partial \epsilon}{\partial t} + u\,\frac{\partial \epsilon}{\partial x} + v\,\frac{\partial \epsilon}{\partial y}\right] = \frac{\partial}{\partial y}\left\{(\mu + \frac{\mu_t}{\sigma_\epsilon})\frac{\partial \epsilon}{\partial y}\right\} + \frac{\epsilon}{e}\left\{C_1\,\mu_t\,(\frac{\partial u}{\partial y})^2 - C_2\,\rho\,\epsilon\right\}$$

Determination of C_1, C_2 and σ_ϵ - L26($\frac{18}{20}$)

From the experimental data on decay of homogeneous turbulence behind a grid in a wind-tunnel, it is found that $e \propto t^{-n}$ where, for $t \rightarrow 0$, $1 < \mathrm{n} < 1.2$. In this flow, both production and diffusion are absent and $v = 0$. Hence

$$\frac{De}{Dt} = \frac{\partial e}{\partial t} + u \, \frac{\partial e}{\partial x} = -\epsilon \quad \text{and} \quad \frac{D\epsilon}{Dt} = \frac{\partial \epsilon}{\partial t} + u \, \frac{\partial \epsilon}{\partial x} = -C_2 \, \frac{\epsilon^2}{e}$$

Simultaneous solution gives $C_2 = (n + 1)/n$. Therefore, taking n = 1.1 (say), $C_2 = 1.91$.

To determine C_1, consider inner turbulent layer where conv = 0, $\nu_t >> \nu$ and production $\nu_t \, (\partial u / \partial y)^2 = \epsilon$ dissipation. Hence ϵ Eqn will read as

$$\frac{\partial}{\partial y} \left\{ \frac{\mu_t}{\sigma_\epsilon} \, \frac{\partial \epsilon}{\partial y} \right\} = \frac{\epsilon^2}{e} \, (C_2 - C_1)$$

Continued ... - L26($\frac{19}{20}$)

In the inner layer, logarithmic law $u^+ = \ln(E\,y^+)/\kappa$ gives

$$\frac{\partial u}{\partial y} = \frac{u_\tau}{\kappa\, y} = \frac{\tau_w}{\rho\,\nu_t} = \frac{u_\tau^2}{\nu_t} \quad \rightarrow \quad \nu_t = u_\tau\,\kappa\,y$$

Hence, since Prod = Diss

$$\epsilon = \nu_t\,(\frac{\partial u}{\partial y})^2 = \frac{u_\tau^3}{\kappa\,y} \quad \rightarrow \quad \frac{\partial \epsilon}{\partial y} = -\,\frac{u_\tau^3}{\kappa\,y^2}$$

Therefore ϵ Eqn becomes

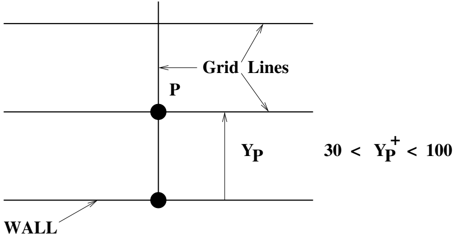
$$\frac{\partial}{\partial y}\left\{\frac{u_\tau \kappa\,y}{\sigma_\epsilon} \times (-\,\frac{u_\tau^3}{\kappa\,y^2})\right\} \quad = \quad \frac{u_\tau^4}{\sigma_\epsilon\,y^2} = \frac{u_\tau^6}{\kappa^2\,y^2\,e}\,(C_2 - C_1)$$

$$\text{Or, } \frac{\kappa^2}{\sigma_\epsilon} = (C_2 - C_1)\,\frac{u_\tau^2}{e} \quad \rightarrow \quad C_1 = C_2 - \frac{\kappa^2}{\sigma_\epsilon\,\sqrt{C_D}} = 1.44$$

where σ_ϵ = 1.3 (from num. comp.), C_D = 0.09 and κ = 0.41

Wall-Function BCs - L26($\frac{20}{20}$)

1. Wall BCs are given at 1st grid node in the inner turb layer
2. Then, $\tau_w = \mu_{eff} \, (\partial u / \partial y)_p = \mu_{eff} \, (u_p / y_p)$. Hence, $\mu_{eff} / y_p = (\rho \, u_\tau \, \kappa) / \ln \, (E \, y_p^+)$ where $u_\tau = C_D^{0.25} \, e_p^{0.5}$.
3. In e-Eqn **Prod** = $\tau_w \, (\partial u / \partial y)_p = \mu_{eff} \, (u_p / y_p)^2$
4. $\bar{\epsilon}_p = y_p^{-1} \int_0^{y_p} \epsilon \, dy$
   $= y_p^{-1} \int_0^{y_p} (\tau_w / \rho)(\partial u / \partial y) \, dy$
   $= (u_\tau^2 \, u_p) / y_p$
   $= C_D^{0.75} \, e_p^{1.5} \, \ln \, (E \, y_p^+) / (\kappa \, y_p)$



Hence, BCs are effected as

$$\text{Source}_e = \mu_{eff} \, (u_p / y_p)^2 - \rho \, \bar{\epsilon}_p$$

$$\epsilon_p = \frac{C_D^{0.75} \, e_p^{1.5}}{\kappa \, y_p}$$

Gives computational economy.