ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-19 LAMINAR DEVELOPING HEAT TRANSFER IN DUCTS

LECTURE-19 LAMINAR DEVELOPING HEAT TRANSFER

1. Importance of Prandtl Number
2. Simultaneous Development of Flow and Heat Transfer for Pr $\simeq 1$
3. Fully Developed Flow - Thermal Entry Length for Pr $>> 1$
4. Slug Flow - Thermal Entry Length for Pr $<< 1$

Importance of Pr - L19($\frac{1}{20}$)

1. In the entrance length of a duct , the velocity and temperature boundary layers develop simultaneously in the presence of heat transfer.
2. For $\text{Pr} \simeq 1$ the two layers can be expected to develop at almost the same rate.
3. However, for $\text{Pr} >> 1$ (Oils) , the temperature boundary layers will develop at a very slow rate, so much so that the velocity profile will already be fully-developed over greater part of thermal development.
4. Conversely, for $\text{Pr} << 1$ (Liquid Metals) , the temperature boundary layer will develop so rapidly that the velocity profile may be assumed to be almost uniform = $\overline{u}$.

Simultaneous Development - L19($\frac{2}{20}$)

Consider entry region of flow between parallel plates 2b apart. Then, the governing equations are

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$\frac{\partial (u^*\, u^*)}{\partial x^*} + \frac{\partial (u^*\, v^*)}{\partial y^*} = -\frac{d\, p^*}{d\, x^*} + \frac{1}{Re}\left[\frac{\partial^2 u^*}{\partial {y^*}^2}\right]$$

$$\frac{\partial (u^*\, T)}{\partial x^*} + \frac{\partial (v^*\, T)}{\partial y^*} = \frac{1}{Re\, Pr}\left[\frac{\partial^2 T}{\partial {y^*}^2} + \frac{\partial^2 T}{\partial {x^*}^2}\right]$$

where $u^* = \frac{u}{\overline{u}}, v^* = \frac{v}{\overline{u}}, p^* = \frac{p}{\rho\, \overline{u}^2}$ $\quad x^* = \frac{x}{D_h}, y^* = \frac{y}{D_h}$

$$Re = \frac{\overline{u} D_h}{\nu} \qquad D_h = 4\, b$$

For $RePr \geq 100$ $\quad \frac{\partial^2 T}{\partial {x^*}^2} << \frac{\partial^2 T}{\partial {y^*}^2}$

Velocity Solution - L19($\frac{3}{20}$)

From Lecture 14,

$$
\begin{aligned}
u' &= u^* + \frac{Re}{\beta^2}\,\frac{d\,p^*}{d\,x^*} \\
&= = C_1 \exp\left(\beta\, y^*\right) + C_2 \exp\left(-\beta\, y^*\right) \\
C_1 &= \frac{(\,Re/\beta^2)\,(d\,p^*\,/\,d\,x^*)}{1 + exp\,(\beta/2)} \\
C_2 &= C_1\; exp\,(\beta/2) \\
v^* &= -\frac{d}{dx^*}\left[\int_0^{y^*} u^*\, dy^*\right]
\end{aligned}
$$

Therefore, the temperature Eqn can be solved by method of linearisation. The method is very cumbersome[1]. Hence only solutions are given.

[1] Heaton H S, Reynolds W C and Kays W M, Int Jnl H & M Transfer, vol 7, p 763, (1964)

Parallel Plates - q_{top} = const - - L19($\frac{4}{20}$)

Top wall receives axially uniform heat flux q_h. Bottom wall is insulated. $x^+ = x^*/(RePr)$, $\theta = (T - T_i)/(q_h\, D_h/k)$, $\theta_b = 2\,x^+$, $Nu_h = h_{h,x}\, D_h/k = 1./\Delta\theta \rightarrow \quad \Delta\theta = (\theta_w - \theta_b)$

parallel plates								
Pr	x^+	.001	.0025	.005	.01	.05	.10	∞
10	Nu_h	15.56	11.46	9.2	7.49	5.55	5.4	5.39
	$\Delta\theta_h$	.064	.087	.11	.134	.18	.185	.186
	$\Delta\theta_{uh}$	-.002	-.005	-.01	-.02	-.059	-.064	-.0643
.7	Nu_h	18.5	12.6	9.62	7.68	5.55	5.4	5.39
	$\Delta\theta_h$	.054	.079	.104	.13	.18	.185	.186
	$\Delta\theta_{uh}$	-.002	-.005	-.01	-.02	-.059	-.064	-.0643
.01	Nu_h	24.2	15.8	11.7	8.80	5.77	5.53	5.39
	$\Delta\theta_h$	.041	.063	.086	.114	.173	.181	.186
	$\Delta\theta_{uh}$	-.002	-.005	-.01	-.02	-.066	-.068	-.064
	θ_b	.002	.005	.01	.02	.10	.2	∞

Circular Tube - q_w = const - L19($\frac{5}{20}$)

u^* and v^* from Langhaar Soln - Uniform heat flux q_w - $\theta_b = 4\,x^+$, $Nu_x = h_x\,D_h/k = 1./\Delta\theta \rightarrow \quad \Delta\theta = (\theta_w - \theta_b)$

Circular Tube								
Pr	x^+	.001	.0025	.005	.01	.05	.10	∞
	Nu_x	14.34	9.93	7.87	6.32	4.51	4.38	4.36
10	$\Delta\theta$	.0697	.1007	.1271	.1582	.222	.228	.229
	Nu_x	17.84	12.08	9.12	7.14	4.72	4.41	4.36
.7	$\Delta\theta$	.0561	.0828	.1096	.1401	.212	.227	.229
	Nu_x	24.2	16.	12.	9.1	6.08	5.73	4.36
.01	$\Delta\theta$	.0413	.0625	.0833	.11	.165	.175	.229
	θ_b	.004	.010	.020	.040	.20	.4	∞

For both parallel plates (pp) and circular tube (ct), thermal development length is $L_h/D_h \simeq 0.1 \times Re\,Pr$. This is *typical for ducts of nearly all cross-sections*. Recall that $L_{flow}/D_h|_{pp} \simeq 0.01 \times Re$ and $L_{flow}/D_h|_{ct} \simeq 0.05 \times Re$.

Parallel Plates (T_w = const) - L19($\frac{6}{20}$)

Here, both plates are held at constant temperature.

$Pr = 5.0$			$Pr = 2.5$			$Pr = 0.7$		
x^+	Nu_x	θ_b	x^+	Nu_x	θ_b	x^+	Nu_x	θ_b
1e-4	40.9	.946	1e-4	56.1	.952	3.6e-4	38.9	.897
3e-4	22.1	.925	2e-4	30.9	.918	7.1e-4	18.4	.840
7e-4	15.2	.905	6e-4	16.8	.888	2.1e-3	11.3	.776
.0012	12.2	.88	.0014	12.1	.857	5e-3	9.05	.705
.003	9.4	.813	.004	8.95	.771	8.6e-3	8.17	.616
.0065	8.2	.715	.006	8.29	.714	.0143	7.79	.516
.009	7.9	.658	.009	7.91	.643	.0321	7.59	.295
.012	7.7	.594	.013	7.71	.565	.0643	7.57	.125
.027	7.6	.374	.024	7.59	.399	.086	7.57	.071
∞	7.54	0.0	∞	7.54	0.0	∞	7.54	0.0

Circular Tube (T_w = const) - L19($\frac{7}{20}$)

	$Pr = 0.7$		$Pr = 2.0$		$Pr = 5.0$	
x^+	Nu_x	Nu_m	Nu_x	Nu_m	Nu_x	Nu_m
.001	16.8	30.6	14.8	25.2	13.5	22.1
.002	12.6	22.1	11.4	19.1	10.6	16.8
.004	9.6	16.7	8.8	14.4	8.2	12.9
.006	8.25	14.1	7.5	12.4	7.1	11.0
.01	6.8	11.3	6.2	10.2	5.9	9.2
.02	5.3	8.7	5.0	7.8	4.7	7.1
.05	4.2	6.1	4.1	5.6	3.9	5.1
∞	3.66	3.66	3.66	3.66	3.66	3.66

$$Nu_m = \frac{1}{x} \int_0^x Nu_x \, dx$$

Thermal Entry Length - L19($\frac{8}{20}$)

For $Pr >> 1$, over greater part of thermal development, the velocity profile can assumed to be fully developed. Hence,
For Parallel Plates

$$u_{fd}\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial y^2}$$
$$\frac{u_{fd}}{\overline{u}} = \frac{3}{2}\left\{1-(\frac{y}{b})^2\right\}$$

For Circular Tube

$$u_{fd}\frac{\partial T}{\partial x} = \frac{\alpha}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r})$$
$$\frac{u_{fd}}{\overline{u}} = 2\left\{1-(\frac{r}{R})^2\right\}$$

BCs at y,r = 0 (symmetry) and y=b and r=R (wall) must be given. Initial condition: T = T_i at x = 0.

Parallel Plates - T_w = const - L19($\frac{9}{20}$)

Governing Eqn

$$\frac{3}{8}\left(1 - {y^*}^2\right)\frac{\partial \theta}{\partial x^*} = \frac{\partial^2 \theta}{\partial {y^*}^2}$$

$$\theta = \frac{T - T_w}{T_i - T_w} \quad , \quad x^* = \frac{(x/b)}{Re\,Pr}, \quad y^* = \frac{y}{b}$$

$$\text{BC} \;\; \theta\left(x^*, 1\right) = 0, \qquad \frac{\partial \theta}{\partial y^*}\,\Big|_{x^*,0} = 0$$

$$\text{IC} \;\; \theta\left(0, y^*\right) = 1.0$$

This is known as the Graetz Problem. It is solved by the Method of separation of variables.

Soln - 1 - T_w = const - L19($\frac{10}{20}$)

Let $\theta = X(x^*) \times Y(y^*)$. Then, substitution gives two ODEs

$$X' + \frac{8}{3}\lambda^2 X = 0 \quad \text{with} \qquad X(0) = 1$$
$$Y'' + \lambda^2 (1 - {y^*}^2)\, Y = 0 \quad \text{with} \qquad Y(1) = Y'(0) = 0$$

The soln for this **Sturm-Louville** Eqn-set is

$$\theta(x^*, y^*) = \sum_{n=0}^{\infty} C_n \exp\left(-\frac{8}{3}\lambda_n^2 x^*\right) \times Y_n(y^*)$$
$$C_n = \frac{\int_0^1 (1 - {y^*}^2)\, Y_n\, dy^*}{\int_0^1 (1 - {y^*}^2)\, Y_n^2\, dy^*} = \frac{-2/\lambda_n}{(d\, Y_n / d\, \lambda_n)_{y^*=1}}$$

λ_n are obtained by integrating Y-Eqn by *shooting method* for various values of λ. Correct values of λ_n correspond to Y(1)=0.

Soln - 2 - T_w = const - L19($\frac{11}{20}$)

$$Nu_x = \frac{h\,(4b)}{k} = -4\,(\frac{\theta^{'}\,(1)}{\theta_b})$$

$$\theta_b = \frac{3}{2}\int_0^1 \theta\,(1 - {y^*}^2)\,dy^*$$

$$= \frac{3}{2}\sum_{n=0}^{\infty}\frac{A_n}{\lambda_n^2}\,exp(-\,\frac{8}{3}\,\lambda_n^2\,x^*)$$

$$\theta^{'}\,(1) = -\sum_{n=0}^{\infty} A_n\,exp(-\,\frac{8}{3}\,\lambda_n^2\,x^*) \rightarrow A_n = -\,C_n\,Y_n^{'}\,(1)$$

$$Nu_x = \frac{8}{3}\left[\frac{\sum_{n=0}^{\infty} A_n\,exp(-\,\frac{8}{3}\,\lambda_n^2\,x^*)}{\sum_{n=0}^{\infty}(A_n/\lambda_n^2)\,exp(-\,\frac{8}{3}\,\lambda_n^2\,x^*)}\right]$$

$$Nu_m = \frac{1}{x^*}\int_0^{x^*} Nu_x\,d\,x^* = -\frac{\ln\theta_b}{x^*}$$

Soln - 3 - T_w = const - L19($\frac{12}{20}$)

Eigen Values and Constants

n	λ_n	$C_n/2$	$A_n/2$
0	1.6816	0.6002	0.85808
1	5.6696	-0.1503	0.56946
2	9.6682	0.08041	0.47606
3	13.6677	-0.05161	0.42397
4	17.6674	0.03982	0.3891
$n > 4$	4n + 5/3	$(-1)^n 1.1356\lambda_n$	$1.0128\lambda_n^{-1/3}$

These values also apply to circular tube[2]

[2]Brown G. M. AIChE,vol 6, p 179-183, (1960)

Soln - 4 - T_w = const - L19($\frac{13}{20}$)

$x^*/4$	θ_b	Nu_x	Nu_m
0	1.0	∞	∞
0.0001	0.9842	26.56	39.736
0.0005	0.95425	15.83	23.416
0.001	0.92774	12.822	18.752
0.003	0.85137	9.5132	13.409
0.005	0.79258	8.5166	11.623
0.01	0.67503	7.7405	9.8249
0.02	0.49804	7.5495	8.7133
0.05	0.20148	7.5407	8.0103
0.10	0.04459	7.5407	7.7755
0.20	0.00218	7.5407	7.6581
∞	0.0	7.5407	7.5407

$Nu_{fd} = (8 / 3) \times \lambda_0^2 = 7.5407$

Parallel Plates - q_w = const - L19($\frac{14}{20}$)

In this case, we define

$$\begin{aligned}
\Psi(x,y) &= \frac{T(x,y) - T_{fd}(x,y)}{q_w\, b \,/\, k} + \frac{T_{fd}(x,y) - T_i}{q_w\, b \,/\, k} \\
&= \theta(x,y) + \theta_{fd}(x,y) \\
\frac{d\,\theta_{fd}}{d\,x^*} &= 4 \quad \rightarrow x^* = \frac{(x/b)}{Re\,Pr}
\end{aligned}$$

Then, we have two equations.

$$\begin{aligned}
\frac{3}{2}\left(1 - {y^*}^2\right) &= \frac{\partial^2 \theta_{fd}}{\partial {y^*}^2} \quad \text{(fully developed part)} \\
\frac{3}{8}\left(1 - {y^*}^2\right)\frac{\partial \theta}{\partial x^*} &= \frac{\partial^2 \theta}{\partial {y^*}^2} \quad \text{(developing part)}
\end{aligned}$$

Soln - 1 - q_w = const - L19($\frac{15}{20}$)

Fully Developed part - Integration gives

$$\theta_{fd} = \frac{3}{4}\,({y^*}^2 - \frac{{y^*}^4}{6}) + 4\,x^* - \frac{39}{280}$$

Developing part -

$$\theta = \sum_{n=1}^{\infty} C_n\, Y_n\,(y^*)\,\exp\,(-\frac{8}{3}\,\lambda_n^2\,x^*)$$

$$C_n = -\frac{\int_0^1 \theta_{fd,(x^*=0)}\,(1 - {y^*}^2)\,Y_n\,(y^*)\;d\,y^*}{\int_0^1 (1 - {y^*}^2)\,Y_n^2\,(y^*)\;d\,y^*}$$

Soln - 2 - q_w = const - L19($\frac{16}{20}$)

Complete solution

$$\Psi = \frac{3}{4}({y^*}^2 - \frac{{y^*}^4}{6}) + 4\,x^* - \frac{39}{280}$$
$$+ \sum_{n=1}^{\infty} C_n\, Y_n\,(y^*)\exp\,(-\frac{8}{3}\,\lambda_n^2\,x^*)$$
$$\Psi_w = \frac{17}{35} + 4\,x^* + \sum_{n=1}^{\infty} B_n \exp\,(-\frac{8}{3}\,\lambda_n^2\,x^*)$$
$$\Psi_b = 4\,x^* \qquad \rightarrow B_n = C_n\,Y_n\,(1)$$
$$Nu_x = \frac{h\,D_h}{k} = (\frac{q_w}{T_w - T_b})(\frac{4\,b}{k}) = \frac{4}{\Psi_w - \Psi_b}$$
$$\frac{1}{Nu_x} = \frac{1}{4}\left[\frac{17}{35} + \sum_{n=1}^{\infty} B_n \exp\,(-\frac{8}{3}\,\lambda_n^2\,x^*)\right]$$

Soln - 3 - q_w = const - L19($\frac{17}{20}$)

Eigen values			Nu values		
n	λ_n	- B_n	$x^*/4$	Nu_x	Nu_m
1	4.2872	0.2222	0.0001	32.153	48.11
2	8.3037	0.07253	0.0005	19.113	28.33
3	12.3106	0.03737	0.001	15.427	22.65
4	16.3145	0.02328	0.005	9.9878	13.89
5	20.3171	0.01611	0.01	8.8031	11.58
6	24.319	0.01192	0.03	8.2458	9.446
7	28.3203	0.00923	0.05	8.2355	8.963
8	32.3214	0.0074	0.10	8.2353	8.599
9	36.3223	0.00609	0.20	8.2353	8.417
10	40.3231	0.00511	∞	8.2353	8.2353

For $n > 10$, $\lambda_n = 4\,n + 1/3$ and - $B_n = 2.401006\,\lambda_n^{-5/3}$

Thermal Entry Length - L19($\frac{18}{20}$)

For $Pr << 1$, over greater part of thermal development, the velocity profile hardly changes. Hence,
For Parallel Plates the governing equation is

$$\overline{u}\,\frac{\partial T}{\partial x} = \alpha\,\frac{\partial^2 T}{\partial y^2}$$

or

$$\frac{1}{4}\,\frac{\partial \theta}{\partial x^*} = \frac{\partial^2 \theta}{\partial {y^*}^2} \quad \rightarrow \theta = \frac{T - T_i}{T_w - T_i} \quad \rightarrow x^* = \frac{(x/b)}{Re\,Pr}$$

where it is assumed that RePr > 100. Then, this parabolic equation can be solved by method of separation of variables using the appropriate boundary conditions.

Parallel Plates - Pr $<<$ 1 - L19($\frac{19}{20}$)

For T_w = const , the soln is

$$\theta = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \left\{ \frac{(2n+1)\,\pi\, y^*}{2} \right\} \times \exp\left(-\pi^2 (2n+1)^2 x^*\right)$$

$$\theta_b = \int_0^1 \theta\, dy^* = \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{\exp\left(-\pi^2 (2n+1)^2 x^*\right)}{(2n+1)^2}$$

$$\frac{\partial \theta}{\partial y^*}\Big|_{y^*=1} = -2 \sum_{n=0}^{\infty} \exp\left(-\pi^2 (2n+1)^2 x^*\right)$$

$$Nu_x = -4\left(\frac{\partial \theta}{\partial y^*}\Big|_{y^*=1}\right) \times \theta_b^{-1}$$

For large x^* $\quad Nu_{fd} \rightarrow \pi^2 = 9.87 \;>\; 7.545$ (for Pr $>> 1$)

Parallel Plates - Pr $<<$ 1 - L19($\frac{20}{20}$)

For q_w = const , the soln is

$$
\begin{aligned}
\Psi &= \theta + \theta_{fd} \\
&= -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(n\,\pi\,y^*\right)\,\exp\left(-4\,\pi^2\,n^2\,x^*\right) \\
&+ \frac{{y^*}^2}{2} + 4\,x^* - \frac{1}{6} \\
\Psi_w &= -\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\,\exp\left(-4\,\pi^2\,n^2\,x^*\right) + 4\,x^* + \frac{1}{3} \\
\Psi_b &= 4\,x^* \\
Nu_x &= 12\left\{1 - \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\,\exp\left(-4\,\pi^2\,n^2\,x^*\right)\right\}^{-1}
\end{aligned}
$$

For large x^* $\quad Nu_{fd} \to 12 \; > 8.235$ (for Pr $>> 1$)