ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-17 FULLY-DEVELOPED LAMINAR FLOW HEAT TRANSFER-1

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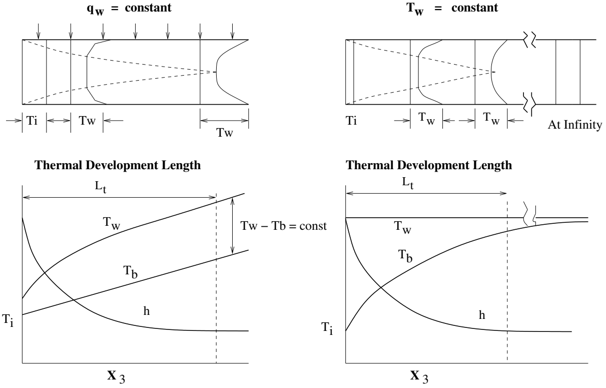
1. Definition of Fully Developed Heat Transfer
2. Nusselt number - Circular Tube family
    1. Circular Tube - Const Wall Heat Flux
    2. Annulus - Const Wall Heat Flux
    3. Circular Tube - Const Wall Temperature
    4. Circular Tube - Const Wall Heat Flux with Viscous Heating
    5. Circular Tube - Circumferential Heat Flux Variation.

FD Heat Tr - Definition - 1 - L17($\frac{1}{19}$)



The figure shows axial variations of wall temperature T_w, fluid-bulk temperature T_b and heat transfer coefficient h in a duct following entry of uniform temperature fluid. Fully developed heat transfer is identified with constancy of h with axial distance

FD Heat Tr - Definition - 2 - L17($\frac{2}{19}$)

We define

$$\Phi(x,r) = \frac{T_w(x) - T(x,r)}{T_w(x) - T_b(x)} \quad \text{where}$$

$$T_b = \frac{\int_A \rho\, c_p\, u\, T\, dA}{\int_A \rho\, c_p\, u\, dA}$$

In Fully-developed heat transfer $\partial\Phi/\partial x$ = 0 or, Φ is constant with x. Therefore,

$$\frac{\partial \Phi}{\partial r}\Big|_{r=R} = -\frac{(\partial T/\partial r)_{r=R}}{T_w(x) - T_b(x)} = \frac{q_w(x)/k}{T_w(x) - T_b(x)} = \frac{h}{k} = \text{constant}$$

In developing heat transfer, however, $\partial\Phi/\partial x$ = f (x, r).

Circular Tube - q_w = const - L17($\frac{3}{19}$)

In fully developed flow and heat transfer, the governing equation will read as

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) = \frac{u}{\alpha}\frac{\partial T}{\partial x} = \frac{u_{fd}}{\alpha}\frac{dT}{dx}, \quad u_{fd} = 2\,\overline{u}\,(1-\frac{r^2}{R^2})$$

$$\text{But } \frac{dT}{dx} = \frac{dT_w}{dx} = \frac{dT_b}{dx} = \frac{q_w\,2\,\pi\,R}{\rho\,c_p\,\overline{u}\,\pi\,R^2} = \text{const}$$

$$\text{or} \quad \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) = 4\,(1-\frac{r^2}{R^2})\,\frac{q_w}{k\,R} \quad (\text{ a1 })$$

with boundary conditions $T = T_w$ at r = R and $\partial T/\partial r$ = 0 at r = 0. Therefore, integrating Equation (a1) twice and using BCs to determine integration constants, we have (next slide)

Circular Tube - q_w = const - L17($\frac{4}{19}$)

$$T = (T_w - \frac{3}{4}\,\frac{q_w}{k\,R}) + \frac{q_w}{k\,R}\,(r^2 - \frac{r^4}{4\,R^4}) \quad \text{Hence,}$$

$$T_b = \frac{\int_A \rho\, c_p\, u\, T\, dA}{\int_A \rho\, c_p\, u\, dA} = \frac{\int_0^R u\, T\, r\, dr}{\int_0^R u\, r\, dr} = T_w - \frac{11}{24}\,(\frac{q_w}{k\,R})$$

$$Nu_D = \frac{h\,D}{k} = (\frac{2\,R}{k})\,\frac{q_w}{T_w - T_b} = \frac{48}{11} = 4.3636$$

Similar analysis for FD flow and heat transfer between two parallel plates separated by distance 2b between the plates gives

$$Nu_{D_h} = \frac{h\,4b}{k} = 8.235$$

Annulus - L17($\frac{5}{19}$)

The governing equation will read as

$$\frac{u}{\alpha}\,\frac{\partial T}{\partial x} = \frac{u_{fd}}{\alpha}\,\frac{dT}{dx} = \frac{1}{r}\,\frac{\partial}{\partial r}\,(r\,\frac{\partial T}{\partial r})$$

$$\frac{u_{fd}}{\overline{u}} = \frac{2}{M}\left[1-(\frac{r}{r_o})^2 + B\ln(\frac{r}{r_o})\right]$$

$$B \equiv \frac{(r^*)^2-1}{\ln r^*}, \quad M \equiv 1+(r^*)^2 - B, \quad r^* \equiv \frac{r_i}{r_o}$$

$$\frac{dT}{dx} = \frac{dT_b}{dx} = \frac{2\,\pi\,(r_o\,q_{w,o} + r_i\,q_{w,i})}{\rho\,c_p\,\overline{u}\,\pi\,(r_o^2 - r_i^2)} \quad (\text{ Heat Balance })$$

Case 1 BCs: $T_{r_i} = T_{w,i}$ and $q_{w,o} = k\,\frac{\partial T}{\partial r}\,|_{r_o}$

Case 2 BCs: $T_{r_o} = T_{w,o}$ and $q_{w,i} = -\,k\,\frac{\partial T}{\partial r}\,|_{r_i}$

where subscripts i and o refer to inner and outer radius.

Annulus Solution - 1 - L17($\frac{6}{19}$)

Integrating twice, we get

$$T = A\left[\frac{r^2}{4} - \frac{1}{16}\frac{r^4}{r_o^2} + B\,\frac{r^2}{4}\left\{\ln\left(\frac{r}{r_o}\right) - 1\right\}\right] + C_1 \ln r + C_2$$

$$A = \frac{4}{M}\left(\frac{q_{w,o}}{k\,r_o}\right)\left(\frac{1 + q^*\,r^*}{1 - (r^*)^2}\right), \quad q^* = \frac{q_{w,i}}{q_{w,o}}$$

Case 1

$$C_1 = -\frac{q_{w,o}\,r_o}{k}\left[q^*\,r^* + \frac{(r^*)^2}{M}\left(\frac{1 + q^*\,r^*}{1 - (r^*)^2}\right)\left((r^*)^2 - B\right)\right]$$

$$C_2 = T_{w,o} - \frac{A\,r_o^2}{4}\left(\frac{3}{4} - B\right) - C_1 \ln(r_o)$$

Case 2

$$C_1 = \frac{q_{w,o}}{k\,r_o} - \frac{A\,r_o^2}{4}(1 - B)$$

$$C_2 = T_{w,i} - \frac{A\,r_i^2}{4}\left[1 - \frac{(r^*)^2}{4} + B\left(\ln(r^*) - 1\right)\right] C_1 \ln(r_i)$$

Annulus Solution - Case 1 - L17($\frac{7}{19}$)

In more compact form

$$
\begin{aligned}
\frac{T - T_{w,o}}{q_{w,o}\, r_o/k} &= \frac{1}{M} \times \frac{1 + q^*\, r^*}{1 - (r^*)^2} \times F_1 - F_2 \\
F_1 &= B - \frac{3}{4} + (\frac{r}{r_o})^2 \left\{ 1 + B\,(\ln(\frac{r}{r_o}) - 1) \right\} - \frac{1}{4}(\frac{r}{r_o})^4 \\
F_2 &= q^*\, r^* + \frac{(r^*)^2}{M} \times \frac{1 + q^*\, r^*}{1 - (r^*)^2} \times \{(r^*)^2 - B\}
\end{aligned}
$$

We define

$$
Nu_o = \frac{h_o\, D_h}{k} = \frac{q_{w,o}\, r_o/k}{T_{w,o} - T_b} \times 2\,(1 - r^*)
$$

where T_b is evaluated by numerical integration.

Annulus Solution - Case 2 - L17($\frac{8}{19}$)

Similarly,

$$\frac{T - T_{w,i}}{q_{w,i}\, r_o/k} = \frac{(r^*)^2}{M} \times \left\{ \frac{1/q^* + r^*}{1 - (r^*)^2} \right\} \times F_3 + \left[\frac{1}{q^*} - \frac{1}{M} \times \left\{ \frac{1/q^* + r^*}{1 - (r^*)^2} \right\} \right] \times \ln\left(\frac{r}{r_i}\right)$$

$$F_3 = \left(\frac{r}{r_i}\right)^2 - \frac{1}{4}\left(\frac{r}{r_i}\right)^2 \left(\frac{r}{r_o}\right)^2 + B \left(\frac{r}{r_i}\right)^2\Big) \left\{ \ln\left(\frac{r}{r_o}\right) - 1 \right\} - 1 + \left(\frac{r^*}{2}\right)^2 - B\, \{\ln(r^*) - 1\}$$

We define

$$Nu_i = \frac{h_i\, D_h}{k} = \frac{q_{w,i}\, r_o/k}{T_{w,i} - T_b} \times 2\,(1 - r^*)$$

where T_b is evaluated by numerical integration.

Annulus Solutions - L17($\frac{9}{19}$)

It is possible to display solutions as

$$Nu_i = \frac{Nu_{ii}}{1 - \theta_i / q^*} \qquad Nu_o = \frac{Nu_{oo}}{1 - \theta_o \, q^*}$$

where $Nu_{ii} = Nu_i \, (q^* = \infty)$ and $Nu_{oo} = Nu_o \, (q^* = 0)$.

If $q^* = q_{w,i} / q_{w,o} = \theta_i$, $Nu_i = \infty$. This does not imply infinite heat transfer but simply that $T_{w,i} = T_b$. Similarly, if $q^* < \theta_i$, $Nu_i < 0$ which implies negative h_i. But, this is acceptable. These arguments also apply to Nu_o.

Values of Nu_{ii}, Nu_{oo} and influence coefficients θ_i and θ_o are given on the next slide

Annulus Solutions - L17($\frac{10}{19}$)

Annulus Solutions					
r^*	Nu_{ii}	θ_i	Nu_{oo}	θ_o	
0.0	∞	∞	4.364	0.0	circular tube
0.05	17.81	2.183	4.791	0.0293	
0.10	11.906	1.383	4.834	0.0561	
0.20	8.499	0.904	4.882	0.1038	
0.30	7.241	0.712	4.928	0.1454	
0.40	6.584	0.601	4.975	0.1822	
0.50	6.182	0.527	5.033	0.2153	
0.60	5.911	0.474	5.100	0.2455	
0.70	5.720	0.432	5.166	0.2733	
0.80	5.579	0.397	5.233	0.2991	
0.90	5.471	0.369	5.306	0.3233	
1.00	5.385	0.346	5.385	0.346	parallel plates

Flat Plate Problem - L17($\frac{11}{19}$)

Prob: Consider FD vel and temp profiles between parallel plates 5 cm apart. The heat fluxes at the two plates are q_1 = 1 kW/m^2 and q_2 = 5 kW/m^2. Calculate $T_{w,1}$ and $T_{w,2}$ at an axial location where T_b = 30^{o}C. Take k = 0.2 W/m-K

soln:

$$Nu_1 = \frac{h_1 \, D_h}{k} = \frac{Nu_{11}}{1 - \theta_1 / q^*} = \frac{5.385}{1 - 0.346/0.2} = -\,7.377$$

Therefore, $h_1 = -7.377 \times 0.2/(2 \times 0.05) = -\,14.753$ W / m^2-K.
Now, $q_1 = h_1\,(T_{w,1} - T_b)$. Therefore,
$T_{w,1} = 1000/(-\,14.753) + 30 = -\,37.78^o$C.

Similar evaluations at plate 2, give Nu_2 = 5.785, h_2 = 11.57 W / m^2-K and $T_{w,2} = 5000/(\,11.57) + 30 = 462.12^o$C.

Circular Tube - T_w = const - L17($\frac{12}{19}$)

In this case, we define

$$Nu_T = \frac{h\,2\,R}{k} = \frac{\partial T}{\partial r}\,|_R \times \frac{2\,R}{T_w - T_b} = \text{constant}$$

Then, from slide 2 and carrying out heat balance, we have

$$\frac{dT}{dx} = \Phi\,\frac{dT_b}{dx}, \qquad \frac{dT_b}{dx} = (\frac{2\,\alpha}{\overline{u}\,R})\,\frac{\partial T}{\partial r}\,|_R$$

Using above relations, the governing equation becomes

$$\frac{1}{r}\,\frac{\partial}{\partial r}\,(r\,\frac{\partial T}{\partial r}) = \frac{u}{\alpha}\,\frac{\partial T}{\partial x} \quad \text{original eqn}$$

$$\frac{1}{r^*}\,\frac{d}{d\,r^*}\,(r^*\,\frac{d\,\Phi}{d\,r^*}) = -2\,Nu_T\,\Phi\,\left\{1 - (r^*)^2\right\}$$

with $\Phi_{r^*=1} = 0$ and $d\,\Phi/d\,r^*\,|_{r^*=0} = 0$. where r^* = r / R

Circular Tube - T_w = const - Soln - L17($\frac{13}{19}$)

The 2nd order ODE is solved by Shooting Method . The procedure is

1. Assume Nu
2. Solve the ODE on a computer starting with $d\,\Phi/d\,r^*\,|_{r^*=0}$
3. Examine if predicted $\Phi_{r^*=1} = 0$.
4. If not, revise Nu

Analytical soln is also possible. It reads

$$\Phi = \sum_{n=0}^{\infty} C_{2n}\,(r^*)^{2n} \quad \text{with} \quad C_0 = 1, \quad C_2 = -\frac{Nu_T}{2}$$

$$\text{and} \qquad C_{2n} = \frac{Nu_T}{4\,n^2}\,(C_{2n-4} - C_{2n-2})$$

The soln is Nu_T = 3.656. For parallel plates, Nu_T = 7.545.

Circular Tube - Viscous Heating - L17($\frac{14}{19}$)

In highly viscous (Pr $>>$ 1) laminar flows, effect of viscous heating must be accounted. Thus, the governing equation is

$$\frac{u_{fd}}{\alpha}\,\frac{dT}{dx} = \frac{1}{r}\,\frac{\partial}{\partial r}\,(r\,\frac{\partial T}{\partial r}) + \frac{\mu}{k}\,(\frac{\partial u_{fd}}{\partial r})^2 \quad (\text{ a1 })$$

$$u_{fd} = 2\,\overline{u}\,(1 - \frac{r^2}{R^2}) \quad \text{and} \quad \frac{dT}{dx} = \frac{dT_b}{dx} = \text{const}$$

$$(\frac{\partial u_{fd}}{\partial r})^2 = (-\frac{4\,\overline{u}\,r}{R^2})^2 = 16\,\frac{\overline{u}^2\,r^2}{R^4}$$

$$2\,\frac{\overline{u}}{\alpha}\,(1 - \frac{r^2}{R^2})\,\frac{dT_b}{dx} = \frac{1}{r}\,\frac{\partial}{\partial r}\,(r\,\frac{\partial T}{\partial r}) + 16\,\frac{\mu}{k}\,\frac{\overline{u}^2\,r^2}{R^4} \quad (\text{ a2 })$$

$$\text{BCs} = (\frac{\partial T}{\partial r})_{r=0} = 0, \quad \text{and} \quad (\frac{\partial T}{\partial r})_{r=R} = \frac{q_w}{k}$$

Viscous Heating -Soln - 1 - L17($\frac{15}{19}$)

To determine $d\,T_b$/dx, we integrate Equation (a1) from r = 0 to r = R. Then, using BCs, it can be shown that

$$\frac{d\,T_b}{dx} = \frac{2\,q_w\,\alpha}{k\,\overline{u}\,R} + \frac{8\,\mu\,\overline{u}}{\rho\,c_p\,R^2} \quad (\text{ a3 })$$

Hence, Equation (a2) will read as

$$2\,\frac{u_{fd}}{k}\,(\frac{q_w}{\overline{u}\,R} + 4\,\frac{\mu\,\overline{u}}{R^2}) = \frac{1}{r}\,\frac{\partial}{\partial r}\,(r\,\frac{\partial T}{\partial r}) + 16\,\frac{\mu}{k}\,\frac{\overline{u}^2\,r^2}{R^4} \quad (\text{ a4 })$$

Substituting for u_{fd}, we integrate this equation twice to determine the temperature profile (see next slide)

Viscous Heating -Soln - 2 - L17($\frac{16}{19}$)

The solution is

$$T - T_w = 2\,\frac{\overline{u}}{k}\,(\frac{q_w}{\overline{u}\,R} + 4\,\frac{\mu\,\overline{u}}{R^2})\,\left[\frac{r^2}{2} - \frac{r^4}{8\,R^2} - \frac{3\,R^2}{8}\right] - \frac{\mu\,\overline{u}^2}{k}\,(\frac{r^4}{R^4} - 1)$$

Hence, T_b evaluates to

$$\begin{aligned} T_w - T_b &= \frac{11}{48} \times \frac{2\,\overline{u}\,R^2}{k}\,(\frac{q_w}{\overline{u}\,R} + 4\,\frac{\mu\,\overline{u}}{R^2}) - \frac{5}{6}\,(\frac{\mu\,\overline{u}^2}{k}) \\ &= \frac{11}{48}\,(\frac{q_w\,D}{k}) + (\frac{\mu\,\overline{u}^2}{k}) \quad (\text{ a5 }) \end{aligned}$$

Dividing this equation by q_wD/k gives the Nusselt number (see next slide)

Viscous Heating -Soln - 3 - L17($\frac{17}{19}$)

Hence, from Equation (a5)

$$
\begin{aligned}
Nu &= \frac{q_w}{T_w - T_b}\,\frac{D}{k} \\
&= \left[\frac{11}{48} + \frac{\mu\,\overline{u}^2}{q_w\,D}\right]^{-1} \\
&= \frac{192}{44 + 192\,Br} \\
Br &= \frac{\mu\,\overline{u}^2}{q_w\,D} \quad \text{Brinkman Number}
\end{aligned}
$$

If Br = 0, we recover Nu = 4.364.

Circular Tube - Axial conduction - L17($\frac{18}{19}$)

In liquid metals (Pr $<<$ 1) and T_w = const. boundary condition, effect of axial conduction becomes important . The governing equation is

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) + \frac{\partial^2 T}{\partial x^2} = \frac{u_{fd}}{\alpha}\frac{dT}{dx}$$

This 2D equation can be solved by analytical method or by Finite Difference method (FDM). The FDM solutions for different Peclet nos (Pe = Re $\times$ Pr) are

Pe	Nu_{fd}	Pe	Nu_{fd}	Pe	Nu_{fd}
0.1	4.057	1.5	3.96	5.0	3.885
0.5	4.017	2.0	3.91	7.5	3.870
1.0	3.980	3.0	3.896	10.0	3.85

As Pe $\rightarrow$ 0, Nu = 4.364, and as Pe $\rightarrow$ ∞, Nu = 3.667.

Circular Tube - $q_w(\theta)$ - L17($\frac{19}{19}$)

Frequently, heat flux variation is irregular around the circumference (due to radiant heating or wall thickness variation in thin-walled tubes) but axially constant . For this case,

$$\frac{u_{fd}}{\alpha}\frac{d\,T_b}{dx} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}, \qquad \frac{d\,T_b}{dx} = \frac{2\,\overline{q}}{\rho\,c_p\,\overline{u}\,R}$$

$$\text{Bcs} \qquad k\left(\frac{\partial T}{\partial r}\right)_{r=R} = q_w(\theta) \quad \text{and} \quad \left(\frac{\partial T}{\partial r}\right)_{r=0} = 0.$$

This 2D equation can be solved by analytical method or by FDM. For $q_w(\theta) = \overline{q}\,(1 + b\cos\theta)$, the solution is

$$Nu_\theta = \left\{\frac{q_w(\theta)}{T_w(\theta) - T_b}\right\}\left(\frac{2\,R}{k}\right) = \frac{1 + b\cos\theta}{11/48 + 0.5\,b\cos\theta}$$

where b is a parameter. Nu_θ can assume both positive and negative values. For b = 0, Nu_θ = 48/11 = 4.364.