Set - II

Section - 1

1. Stress $\propto$ strain

2. L^2 (L - length)

3. $6-3=3$

4. $\int_{-\infty}^{\infty} g(x)\,\delta(x-5)\,dx = g(5) = 1+5+5^2 = 31$

5. $\psi = xy, \quad u = -x \,;\; v = y$

$$|\bar{u}| = \sqrt{u^2+v^2} = \sqrt{x^2+y^2}$$

$$|\bar{u}|_{(1,1)} = \sqrt{2}$$

6. $\bar{u}\cdot\bar{n} = \bar{U}\cdot\bar{n}$ (no external forces $\Rightarrow \bar{U}=0$)

$\Rightarrow \bar{u}\cdot\bar{n} = 0$

7. for $x=0$, $\bar{n} = \hat{i} = (1,0)$

$$\bar{t} = (0,1)$$

$$\bar{u}\cdot\bar{t} = \bar{U}\cdot\bar{t} \Rightarrow \bar{u}\cdot(0,1) = (3,-2)\cdot(0,1)$$

$$\Rightarrow v = -2$$

8. viscous force

9. center line

10. Darcy equation

SECTION-I

Q11. electric ; Q12. qE ; Q13. 0 , Q14. greater.

Q15. $\nabla \cdot u = 0$, Q16. $\nabla \cdot N_i = 0$, Q17. 0 ;

Q18. $e(z_1 n_1 + z_2 n_2)$, Q19. $\frac{U}{E}$; Q20. low

Section - 2

1. if $\bar{u} = (v_r, 0, 0)$ then equation of continuity $\Rightarrow \frac{\partial}{\partial r}(r v_r) = 0$ 1 mark

$\Rightarrow r v_r = f(z)$ (due to axi-symmetry)

$\Rightarrow v_r = \frac{1}{r} f(z)$ 1 mark

2. $r u_r = -\frac{\partial \psi}{\partial z}$; $r u_z = \frac{\partial \psi}{\partial r}$

$\Rightarrow u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}$; $u_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$

$\Rightarrow -\frac{1}{r}\frac{\partial \psi}{\partial z} = \frac{r^2 z}{3} \Rightarrow \psi = -\frac{r^3 z^2}{6} + f(r)$ 1 mark

$\Rightarrow \frac{\partial \psi}{\partial r} = -\frac{r^2 z^2}{2} + f'(r) = -\frac{r^2 z^2}{2} \Rightarrow f(r) = \text{const.}$

$\Rightarrow \psi = -\frac{r^3 z^2}{6} + \text{const}$ 1 mark

3. volume of the sphere $= \frac{4}{3}\pi r^3$.

$= \frac{4}{3} \times \frac{22}{7} \times 10^3 \text{ cm}^3 = 4190.476 \text{ cm}^3$ 1 mark

porosity $\phi = \frac{\text{volume of the void}}{\text{volume of the sphere}} = \frac{40}{4190.476}$

$= 0.0095$ 1 mark

4. $D_p = 1\,mm = 0.1\,cm$

$$K = \frac{D_p^2 \phi^3}{180(1-\phi)^2} = 4 \times 10^{-11}\ cm^2$$ 2 marks

5. $$T(r,\theta) = \sum_{n=0}^{\infty} \left(a_n r^n + \frac{b_n}{r^{n+1}}\right) P_n(\cos\theta)$$

$$\rightarrow \sum_{n=0}^{\infty} \frac{b_n}{r^{n+1}} P_n(\cos\theta) \text{ as } r \to \infty$$ 1 mark

$$= \frac{50 \cos\theta}{r^2} \Rightarrow b_0 = 0$$

$b_1 = 50$ 1 mark

$b_n = 0, \forall n \geq 2$

6. $G(\bar{x}, \bar{x}_0) = \ln r$, $\bar{x} = (x, y)$

$\bar{x}_0 = (1, 2)$

$$\frac{\partial G}{\partial x} = \frac{\partial G}{\partial r} \frac{\partial r}{\partial x} = \frac{1}{r} \cdot \frac{x}{r} = \frac{x}{r^2},$$ 1 mark

1 mark
$r^2 = (x-1)^2 + (y-2)^2$

7. Stokes equations $\nabla \cdot \bar{u} = 0$

$-\nabla p + \mu \nabla^2 \bar{u} = 0$

i) $\Rightarrow$ $\bar{v}$ biharmonic, ii) $\nabla \times \bar{v}$: harmonic

1 + 1 marks

8. i) p harmonic, ii) ψ : bi-harmonic

1 + 1 marks

9. $\bar{u} = (yz - x, y + z, 0) = (u, v, w)$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \mu z$$

at least 1 correct - 1 mark;
any 2 correct - 2 marks

$$\tau_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \mu$$

$$\tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \mu y$$

10. $u = 2cxy = -\dfrac{\partial \psi}{\partial y} \Rightarrow \psi = -cxy^2 + f(x)$

1 mark

$$\frac{\partial \psi}{\partial x} = -cy^2 + f' = v = -cy^2 + c(a^2 + x^2)$$

$$\Rightarrow \quad f'(x) = c(a^2 + x^2)$$

$$\Rightarrow \quad f(x) = c\left(a^2 x + \frac{x^3}{3}\right) + \text{constant}$$

$$\therefore \psi = -cxy^2 + c\left(a^2 x + \frac{x^3}{3}\right) + \text{constant}$$

1 mark

SECTION-II

Q11. Electric flux, $\oint_S (E\cdot\hat{n})\,ds = \frac{q}{\varepsilon_e}$ — (1M)

$= 7.19\times10^{6}\ V/m$ — (2M)

Q12. $\sigma = \varepsilon_e K \zeta$ — (1M)

$= 1.93\times10^{-3}\ C/m^2$ — (2M)

Q13. $\psi = \zeta \exp(-K x)$ (1M) $= 2.451\times10^{-2}\,V$ — (2M)

Q14. $c_{Na^+} = c_0 \exp\left(-\frac{\psi}{\phi_0}\right)$ — (1M)

$= 0.3876\ mol/m^3$ — (2M)

Q15. $c_{Cl^-} = c_0 \exp\left(-\frac{\psi}{\phi_0}\right)$ — (1M)

$= 2.58\ mol/m^3$ — (2M)

Q16. Charge density, $\rho_e = \cancel{eC_{Na^+}}$ (S-2) P-2

$$N_A e(c_{Na^+} - c_{Cl^-}) - (1M)$$

$$= -2.115 \times 10^{3}\ C/m^3. - (2M)$$

Q17.

Debye length $\kappa^{-1} = \sqrt{\dfrac{\varepsilon \phi_0}{2ez^2 n_0}}$ — (1M)

$$= 0.932 \times 10^{-8}\ m. - (2M)$$

Q18.

~~axial~~ Velocity along the x-axis,

$$u(y) = \frac{h^2 p_x}{2\mu}\left[1 - \left(\frac{y}{h}\right)^2\right] - \frac{\varepsilon E_x J}{\mu}\left[1 - \frac{\cosh(\kappa y)}{\cosh(\kappa h)}\right]$$

— (1M)

$$= -6.88 \times 10^{-7}\ m/s \quad - (2M)$$

Q19. for $\kappa a << 1$, $U_E = \dfrac{2}{3}\dfrac{\varepsilon E_0 J}{\mu}$ — (1M)

$$= 1.2 \times 10^{-6}\ m/s - (2M)$$

Q20. $(1+x_i^2)\dfrac{y_{i+1} - 2y_i + y_{i+1}}{h^2} + 4x_i \dfrac{y_{i+1} - y_{i-1}}{2h} + 2y_i = 2$

$h = 1/3$.

— (1M)

or, $(9 + 9x_i^2 - 6x_i)y_{i-1} + (-9x_i^2 - 7)y_i$

$+ (9 + 9x_i^2 + 6x_i)y_{i+1} = 2.$ — (2M)

Section - 3

1. x-component of the given equation is

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\mu u}{K}$$

using the given definition of non-dim. variables, we show the adjustment

Consider $\rho \frac{\partial u}{\partial t} = \rho \frac{\partial u/U \cdot U}{\frac{\rho L^2}{\mu}\, \partial t / \frac{\rho L^2}{\mu}}$

$$= \rho \frac{\mu U}{\rho L^2} \frac{\partial u'}{\partial t'}$$

Consider $u\frac{\partial u}{\partial x} = U\, u/U \; \frac{U \partial u/U}{L\,\partial x/L}$

$$= \frac{U^2}{L} u' \frac{\partial u'}{\partial x'}$$

Similarly, we do for the other terms, then we have

$$\rho\left(\frac{\mu U}{\rho L^2}\frac{\partial u'}{\partial t'} + \frac{U^2}{L} u'\frac{\partial u'}{\partial x'} + \frac{U^2}{L} v'\frac{\partial u'}{\partial y'}\right)$$

$$= -\frac{\rho U^2}{L}\frac{\partial p'}{\partial x'} + \frac{\mu U}{L^2}\nabla'^2 u' - \frac{\mu U}{K} u'$$

any 3 groups correct - 3 marks

dividing by $\frac{\mu U}{L^2}$, we get

$$\frac{\partial u'}{\partial t'} + \frac{\rho U L}{\mu}\left(u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'}\right)$$

$$= -\frac{\rho U L}{\mu}\frac{\partial p'}{\partial x'} + \nabla'^2 u' - \frac{L^2}{K} u'$$

$$\Rightarrow \frac{\partial u'}{\partial t'} + Re\left(u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'}\right)$$

$$= -Re\frac{\partial p'}{\partial x'} + \nabla^2 u' - \frac{1}{Da} u'$$

$\therefore \alpha = 1;\ \beta = Re;\ \lambda = \frac{1}{Re};\ \gamma = \frac{1}{Da}$ all correct: remaining 2 marks

2. $K = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \Rightarrow K^{-1} = \begin{pmatrix} 1/k_1 & 0 \\ 0 & 1/k_2 \end{pmatrix}$

x-component: $-\frac{\partial p}{\partial x} + \mu \nabla^2 u - \frac{\mu u}{k_1} = 0$ 2 marks

y-component: $-\frac{\partial p}{\partial y} + \mu \nabla^2 v - \frac{\mu v}{k_2} = 0$

Stream function $u = -\frac{\partial \psi}{\partial y};\ v = \frac{\partial \psi}{\partial x}$

eliminating p, we get

$$\nabla^4 \psi - \frac{\mu}{k_1}\frac{\partial^2 \psi}{\partial y^2} - \frac{\mu}{k_2}\frac{\partial^2 \psi}{\partial x^2} = 0$$

correct form: 3 marks; partial correct: 1 marks

(3) $P \rho^a L^b v^c = M^0 L^0 T^0$ 1 mark

$\Rightarrow (ML^{-3})^a L^b (LT^{-1})^c ML^{-1}T^{-2} = M^0 L^0 T^0$ 2 marks

$\Rightarrow a+1=0;\ -3a+b+c=0;\ -c-2=0$

1 mark

$\Rightarrow a=-1,\ c=-2,\ b=0$

$\Rightarrow \pi_1 = P/\rho v^2$ 1 mark

(4.) $\rho^a v^b L^c K = M^0 L^0 T^0$ 1 mark

$\Rightarrow (ML^{-3})^a (LT^{-1})^b L^c L^2 = M^0 L^0 T^0$ 2 marks

$\Rightarrow a=0;\ -3a+b+c+2=0$

$b=0$ 1 mark

$\Rightarrow c=-2$

$\Rightarrow \pi_2 : K/L^2$ 1 mark

SECTION - III

S - 2
P - 3

Q5. Equation for the electric field

$$\nabla\cdot(\varepsilon_e E) = +\rho_e$$

or, $\nabla\cdot(-\varepsilon_e \nabla\phi) = \rho_e$ — (2M)

or, $\nabla^2\phi = -\frac{\rho_e}{\varepsilon_e} = -\frac{ez}{\varepsilon_e}(n_1 - n_2)$ — (3M)

$$n_1 = n_0 \exp\left(-\frac{ze\phi}{k_B T}\right)$$

$$n_2 = n_0 \exp\left(\frac{ze\phi}{k_B T}\right)$$ — (4M)

$\therefore$ $\nabla^2\phi = +\frac{2ez}{\varepsilon_e} n_0 \sinh\left(\frac{ze\phi}{k_B T}\right)$ — (5M)

Q6. $\cancel{u(y)} = \frac{d^2\psi}{dy^2} = -\frac{\rho_e}{\varepsilon_e}$

$$\rho_e = -2zen_0 \sinh\left(\frac{ze\psi}{k_B T}\right)$$

or, $\frac{d^2\psi}{dy^2} = \kappa^2\psi$ where $\kappa = \left(\frac{2n_0 z^2 e^2}{\varepsilon_e k_B T}\right)^{1/2}$

At $y = h, \psi = \zeta$

$y = 0, \frac{d\psi}{dy} = 0$

gives, $\psi = \zeta \frac{\cosh(\kappa y)}{\cosh(\kappa h)}$ — (2M)

$$\rho_e = -\varepsilon_e \kappa^2 \zeta \frac{\cosh(\kappa y)}{\cosh(\kappa h)}$$ S-2 P-4

momentum eqn. in x-direction,

$$\mu \frac{d^2u}{dy^2} = \rho_e \frac{\partial \phi}{\partial x} \qquad \phi = \psi + E_0 \cdot x$$

– (3M)

or, $$\mu \frac{d^2u}{dy^2} = \varepsilon_e \kappa^2 \zeta E_0 \cdot \frac{\cosh(\kappa y)}{\cosh(\kappa h)}$$ – (4M)

$$u = 0 \text{ at } y = h$$

$$\frac{du}{dy} = 0 \text{ at } y = 0$$

Therefore, $$u = -\frac{\varepsilon_e E_0 \zeta}{\mu}\left[1 - \frac{\cosh(\kappa y)}{\cosh(\kappa h)}\right]$$ – (5M)

9F.

$$u_t = u_{xx}$$

Discretize by Cranck-Nicholson scheme:

$$\frac{u_i^{n+1} - u_i^n}{\delta t} = \frac{1}{2}\left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\delta x)^2} + \frac{u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}}{(\delta x)^2}\right]$$

– (2M)

or, $$\frac{\delta t}{2(\delta x)^2} u_{i-1}^{n+1} - \left(1 + \frac{\delta t}{(\delta x)^2}\right) u_i^{n+1} + \frac{\delta t}{2(\delta x)^2} u_{i+1}^{n+1}$$

$$= -\frac{\delta t}{2(\delta x)^2} u_{i-1}^{n} - \left(1 - \frac{\delta t}{(\delta x)^2}\right) u_i^{n}$$

$$- \frac{\delta t}{2(\delta x)^2} u_{i+1}^{n}$$ – (3M).

S - 2
P - 5

$\delta t = \frac{1}{36}$, $\frac{\delta t}{(\delta x)^2} = \frac{1}{4}$

$$u_{i-1}^{n+1} - 10 u_i^{n+1} + u_{i+1}^{n+1} = - u_{i-1}^{n} - 6 u_i^{n} - u_{i+1}^{n}$$ — (4M)

$u_0^1 = 0$, $u_1^1 = 0.6735753$, $u_2^1 = 0.6735753$, $u_3^1 = 0$ — (5M)

Similar marks distribution if a first-order implicit scheme is used.

Q. 8 Equations for fluid flow

$$\nabla \cdot u = 0$$

$$\rho \frac{\partial u}{\partial t} + \rho (u \cdot \nabla) u = - \nabla p + \mu \nabla^2 u + \rho_e \nabla \varphi$$

— (1M + 2M)

Non-dimensional form

$$Re \frac{\partial u}{\partial t} + Re (u \cdot \nabla) u = - \nabla p + \nabla^2 u + \left(\frac{\kappa a}{2}\right)^2 \rho_e \nabla \phi$$

$$\nabla \cdot u = 0$$ —

$Re = \frac{U_0 a \rho}{\mu}$, $\kappa = \sqrt{\frac{2 z_i e n_0}{\epsilon_e \varphi_0}}$, ~~$\rho_e$~~ n_i — (5M)