Set-I

Section - 1

1. boundary layer

2. $ML^{-1}T^{-1}$

3. $(n-r)$

4. $\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \bar{u} \cdot \nabla F = 0$

5. $4\sqrt{2}$

6. surface deformation

7. Stokes equation

8. $(1/2, -1/2)$

9. upper plate

10. $\bar{u} \cdot \bar{n} = 0$

SECTION - I

Q11. $\nabla^2\phi = 0$; Q12. $O(h^2)$; Q13. inversely ; Q14. directly ; Q15 directly ; Q16. directly ;
Q17. directly ; Q18 TRUE ; Q19. TRUE ; Q20. FALSE

Section – 2

1. Let $f = y - 1 - \cos 2\pi x$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{(2\pi \sin \pi x, 1)}{\sqrt{4\pi^2 \sin^2 \pi x + 1}}$$ 1 mark

$$\bar{t} = \frac{(1, -2\pi \sin 2\pi x)}{\sqrt{4\pi^2 \sin^2 \pi x + 1}}$$

no-slip condition $\bar{u}\cdot\bar{t} = \bar{U}\cdot\bar{t}$ at $y = 1 + \cos 2\pi x$

$$\Rightarrow (u, v)\cdot(1, -2\pi \sin 2\pi x)$$
$$= (V, 0)\cdot(1, -2\pi \sin 2\pi x)$$

1 mark

$$\Rightarrow \quad u - v\, 2\pi \sin 2\pi x = V$$

2. $\frac{\partial p}{\partial \theta} = 0 \Rightarrow p = p(r, z)$

$$dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial z} dz = \rho\omega^2 r\, dr - \rho g\, dz$$

$$\Rightarrow p = \rho\omega^2 \frac{r^2}{2} - \rho g z + c$$ 1 mark

$$p(0, h) = 0 \Rightarrow c = \rho g h$$

$$\therefore p(r, z) = \rho\omega^2 \frac{r^2}{2} - \rho g z + \rho g h$$ 1 mark

(3). Stokes drag $F = 6\pi\mu U a$ 1 mark

$= 6 \times \frac{22}{7} \times 2 \times 7 \times 2 \ N$

$= 528$ Newton 1 mark

(4). Given $T(r,\theta) \to \sum_{n=0}^{\infty} a_n r^n P_n(\cos\theta)$ as $r \to \infty$

1 mark $= a_0 + a_1 r\cos\theta + a_2 r^2 \frac{1}{2}(3\cos^2\theta - 1) + \cdots$

$\sim T_\infty + T_\infty r^2(3\cos^2\theta - 1)$

$\Rightarrow a_0 = T_\infty, \quad a_1 = 0$

$\frac{a_2}{2} = T_\infty. \Rightarrow a_2 = 2T_\infty.$ 1 mark

$\therefore a_n = 0 \quad \forall n \geq 3$

(5). Given equation in component form

$0 = -\frac{\partial p}{\partial x} + \mu \nabla^2 u - \frac{\mu u}{K}$ — (1) 1 mark

$0 = -\frac{\partial p}{\partial y} + \mu \nabla^2 v - \frac{\mu v}{K}$ — (2)

$\nabla \cdot \bar{u} = 0 \Rightarrow u = \frac{\partial \psi}{\partial y}, \; v = -\frac{\partial \psi}{\partial y}$ — (3)

(3) in (1) (2) and eliminating pressure, we get

$\nabla^2\left(\nabla^2 - \frac{1}{K}\right)\psi = 0$ 1 mark

(6). Brinkman equation under uni-directional flow, reduces to

$$-\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} - \frac{\mu u}{K} = 0 \quad \text{—(1)}$$

$$-\frac{\partial p}{\partial y} = 0 \quad \text{—(2)}$$

(2) $\Rightarrow$ $p = p(x)$ & given $\frac{dp}{dx} = 0$

$\therefore$ (1) reduces to $\mu \frac{d^2 u}{dy^2} - \frac{\mu u}{K} = 0$ 1 mark

$\left(\because \nabla \cdot \bar{u} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0 \Rightarrow u = u(y)\right)$

$\Rightarrow$ $u(y) = c_1 \cosh(\alpha y) + c_2 \sinh(\alpha y)$, $\alpha^2 = \frac{1}{K}$

b.c.s: $u = 0$ at $y = 0$ $\Rightarrow$ $c_1 = 0$
$u = U$ at $y = h$ $\quad c_2 = \frac{U}{\sinh(\alpha h)}$

$\therefore$ $u(y) = U \frac{\sinh(\alpha y)}{\sinh(\alpha h)}$ 1 mark, (αh is non-zero)

(7). maximum velocity occurs at $y = h$, 1 mark

$\therefore$ $u_{max} = U$ (which is at $y = h$).

1 mark

(8). vol. of the sphere $= \frac{4}{3}\pi \cdot 3^3 = 36\pi \text{ cm}^3$

vol. of the void $= 60 \text{ cm}^3$ 1 mark

porosity $(\phi) = \frac{60}{36\pi} = 0.53$ 1 mark

(9).

$$K = \frac{D_p^2 \phi^3}{180(1-\phi)^2} = \frac{(0.2)^2 (0.53)^3}{180 (1-0.53)^2}$$

$\simeq 1.50 \times 10^{-4} \text{ cm}^2$ 2 marks

(10). $(u, v) = (-x, y+t)$

$-\frac{\partial \psi}{\partial y} = -x \;; \quad \frac{\partial \psi}{\partial x} = y + t$

$\Rightarrow \psi = xy + tx + f(y)$ 1 mark

$\Rightarrow x = x + f'(y) \Rightarrow f = c \text{ (constant)}$

$\therefore \psi = xy + tx + c$

$= xy + 2x + c$ at $t = 2$

1 mark

SECTION–II

Q11. $\lambda = \sqrt{\dfrac{\varepsilon_e \phi_0}{2 e z^2 n_0}}$ – 1M $= \cancel{3.04 \times 10^{-10}\ m}$. 0.932×10^{-8} m – 2M

Q12. $K = 3.28 \times 10^{8}\ m^{-1}$

$\psi(y) = \zeta \dfrac{\cosh(\kappa y)}{\cosh(\kappa h)}$ 1M $= 3.36 \times 10^{-5}$ V. – 2M

Q13. $u(y) = -\dfrac{\varepsilon_e E_x \zeta}{\mu}\left[1 - \dfrac{\cosh(\kappa y)}{\cosh(\kappa h)}\right]$ – 1M

$= -6.94923 \times 10^{-6}$ m/s – 2M

Q14. $Q = -\dfrac{\varepsilon_e E_x \zeta\, 2h}{\mu}\left[1 - \dfrac{\tanh(\kappa h)}{\kappa h}\right]$ – 1M

$= -3.05293 \times 10^{-13}\ m^3/ms$ – 2M

Q15. $$\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}+2\cdot\frac{y_{i+1}-y_{i-1}}{2h}-8y_i=0$$ – 1M

or, $$\left(\frac{1}{h^2}-\frac{1}{h}\right)y_{i-1}+\left(-8-\frac{2}{h^2}\right)y_i+\left(\frac{1}{h^2}+\frac{1}{h}\right)y_{i+1}=0.$$ – 2M

Q16. $h=0.25$,
$$-10y_1+5y_2=-3$$
$$3y_1-10y_2+5y_3=0$$
$$3y_2-10y_3=0$$ – 2M

Q17. when $a/\lambda<<1$, $$U_E=\frac{2}{3}\frac{\varepsilon_e E_0 \zeta}{\mu}$$ – 1M

$$=2.376843\times10^{-5}\,m/s$$ – 2M

Q18. when $a/\lambda>>1$, $$U_E=\frac{\varepsilon_e E_0\zeta}{\mu}$$ – 1M

$$=3.1722\times10^{-5}\,m/s.$$ – 2M

Q19. The governing eqn. for the pressure driven flow, $$\mu\frac{d^2u}{dx^2}=\frac{dp}{dx}$$

or, $$u=-\frac{p_x}{2\mu}y^2+Ay+B$$ where $p_x=\frac{-dp}{dx}$

with, at $y=0$, $u=\beta\frac{du}{dy}$ and $y=h$, $u=0$ – 1M

Therefore $u = -\frac{p_x}{2\mu}(y^2 - h^2) + \frac{p_x h^2}{2\mu(\beta+h)}(y-h)$ – 2M

Q20. $u_{xx} = 32 u_t$

S-1/P-3

or, $u_t = \frac{1}{32} u_{xx}$

or, $\frac{u_i^{n+1} - u_i^n}{\delta t} = \frac{1}{32} \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\delta x)^2}$

or, $u_i^{n+1} = u_i^n + \frac{1}{64}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$ – 1M

$\therefore u_1^1 = 0, \; u_1^2 = 0.001953125$ – 2M

Section - 3

① for $\Pi_1 = f(p, L, V, \mu) = 0$,

$L^a V^b \mu^c p = M^0 L^0 T^0$ 1 mark

$\Rightarrow L^a (LT^{-1})^b (ML^{-1}T^{-1})^c \cdot (ML^{-1}T^{-2}) = M^0 L^0 T^0$

2 marks

$\Rightarrow L^{a+b-c-1} = L^0 \Rightarrow a+b-c = 1$

$T^{-b-c-2} = T^0 \Rightarrow b+c = -2$ 1 mark

$M^{c+1} = M^0 \Rightarrow c = -1$

$\Rightarrow a = 1, \; b = -1, \; c = -1$

$\therefore \Pi_1 : \; p L V^{-1} \mu^{-1} = \dfrac{p}{(\mu V/L)}$ 1 mark

② for $\Pi_2 = f(L, V, \mu, K) = 0$

$L^a V^b \mu^c K = M^0 L^0 T^0$ 1 mark

$\Rightarrow L^a (LT^{-1})^b (ML^{-1}T^{-1})^c L^2 = M^0 L^0 T^0$ 2 marks

$\Rightarrow L^{a+b-c+2} = L^0 \Rightarrow a+b-c = -2$

$T^{-b-c} = T^0 \Rightarrow b = -c$ 1 mark

$M^c = M^0 \Rightarrow c = 0$

$\Rightarrow c = 0, \; b = 0, \; a = -2$

$\Pi_2 : K/L^2$

1 mark

(3) x- component of the given equation is ~~∂u~~

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\nabla^2 u-\frac{\mu u}{K}$$

introducing the non-dimensional variables as given, we have

$$\frac{\rho U^2}{L}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)$$

any 3 gropus correct: 3 marks

$$=-\frac{\mu U}{L^2}\frac{\partial p}{\partial x}+\frac{\mu U}{L^2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-\frac{\mu U}{K}u$$

(primes dropped)

$$\Rightarrow \quad Re\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)$$

$$=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{L^2}{K}u$$

$\therefore\ \alpha = Re;\quad \Lambda = 1;\quad \beta = 1;\quad \gamma = \frac{1}{Da}$ 2 marks

④. $K = \begin{pmatrix} k_1 & k_2 \\ k_2 & \lambda k_1 \end{pmatrix}$, $|K| = \lambda k_1^2 - k_2^2 > 0$

hence K^{-1} exist.

$$\therefore K^{-1} = \frac{1}{|K|}\begin{pmatrix} \lambda k_1 & -k_2 \\ -k_2 & k_1 \end{pmatrix}$$ 1 mark

$$\therefore K^{-1}\bar{u} = \frac{1}{|K|}\begin{pmatrix} \lambda k_1 u - k_2 v \\ -k_2 u + k_1 v \end{pmatrix}$$

$\therefore$ x and y- components are

$$-\frac{\partial p}{\partial x} + \mu \nabla^2 u = \frac{\mu}{|K|}(\lambda k_1 u - k_2 v)$$ 2 + 2 marks

$$-\frac{\partial p}{\partial y} + \mu \nabla^2 v = \frac{\mu}{|K|}(-k_2 u + k_1 v)$$

SECTION – III

Q5. Nerst-Planck eq^n ~~for~~ at steady ~~flow~~ state

$$\nabla \cdot \left(c_i u - D_i \nabla c_i + \frac{z_i D_i F}{RT} c_i E \right) = 0 \quad - (2M)$$

At steady state under no bulk flow i.e. $u = 0$ with zero flux of ions along the normal to the surface., ~~molar~~ ~~concentration~~ ~~of ith~~

Then $$\frac{dc_i}{dx} + \left(\frac{z_i c_i F}{RT} \right) \frac{d\phi}{dx} \quad - (3M).$$

or, $$\frac{d(\ln c_i)}{dx} + \left(\frac{z_i F}{RT} \right) \frac{d\phi}{dx} = 0.$$

Thus, $c_i = c_i^0 \exp\left(-\frac{z_i F}{RT}\phi\right)$ — (5M)

Section - III

Q. 6. (S-1 P-4)

$$\frac{\partial u}{\partial x} = 0, \quad u = u(y)$$

$$\mu \frac{d^2 u}{dy^2} = \rho_e \left(E_0 - \frac{d\phi}{dx}\right)$$

$$\frac{d^2\phi}{dy^2} = -\rho_e/\epsilon_e \quad \text{— (2M)}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y} + \frac{g}{\phi_0}\frac{\partial \phi}{\partial y}\right) = 0$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y} - \frac{f}{\phi_0}\frac{\partial \phi}{\partial y}\right) = 0$$

$$\frac{d^2\phi}{dy^2} = -\frac{(\kappa h)^2}{2}(g-f) \quad \text{— (4M)}$$

$u = 0,\ f = f_0,\ g = g_0,\ \phi = \zeta$ on $y = 0$

— (5M)

Q. 7

$$\frac{u_i^{n+1} - u_i^n}{\delta t} + \frac{1}{2}\left[u\frac{\partial u}{\partial x}\Big|_i^{n+1} + u\frac{\partial u}{\partial x}\Big|_i^n\right] = \frac{\nu}{2}\left[\frac{\partial^2 u}{\partial x^2}\Big|_i^{n+1} + \frac{\partial^2 u}{\partial x^2}\Big|_i^n\right]$$

— (2M)

$$\frac{u_i^{n+1}}{\delta t} + \frac{u_i^{n+1}}{2}\left[\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\delta x}\right] - \frac{\nu}{2}\left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\delta x)^2}\right]$$

$$= \frac{\nu}{2}\left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\delta x)^2}\right] - \frac{u_i^n}{2}\left[\frac{u_{i+1}^n - u_{i-1}^n}{2\delta x}\right] + \frac{u_i^n}{\delta t}$$

— (3M)

At every iteration

$$\left(u_i^{n+1}\right)^{(k+1)} = \left(u_i^{n+1}\right)^{(k)} + \Delta u_i^{n+1} \quad \text{S-1, P-5}$$

$$i = 1, 2, \cdots, N-1; \; k \geq 0$$

k is the iteration index — 4M

Substituting and retaining the linear terms of Δu_i^{n+1}'s

$$\Delta u_{i-1}^{n+1} \left[\left(u_i^{n+1}\right)^{(k)} \cdot \left(-\frac{1}{4\delta x}\right) - \frac{\nu}{2} \cdot \frac{1}{(\delta x)^2} \right]$$

$$+ \Delta u_i^{n+1} \left[\frac{1}{\delta t} + \frac{1}{4\delta x} \left(\left(u_{i+1}^{n+1}\right)^{(k)} - \left(u_{i-1}^{n+1}\right)^{(k)} \right) + \frac{\nu}{(\delta x)^2} \right]$$

$$+ \Delta u_{i+1}^{n+1} \left[\left(u_i^{n+1}\right)^{(k)} \cdot \left(\frac{1}{4\delta x}\right) - \frac{\nu}{2} \cdot \frac{1}{(\delta x)^2} \right]$$

$$= \frac{\nu}{2} \frac{\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}{(\delta x)^2} - \frac{u_i^n}{2} \frac{u_{i+1}^n - u_{i-1}^n}{2\delta x}$$

$$+ \frac{u_i^n}{\delta t} - \frac{\left(u_i^{n+1}\right)^{(k)}}{\delta t} - \frac{\left(u_i^{n+1}\right)^{(k)}}{4\delta x} \left(\left(u_{i+1}^{n+1}\right)^{(k)} - \left(u_{i-1}^{n+1}\right)^{(k)} \right)$$

$$+ \frac{\nu}{2} \left[\frac{\left(u_{i+1}^{n+1}\right)^{(k)} - 2\left(u_i^{n+1}\right)^{(k)} + \left(u_{i-1}^{n+1}\right)^{(k)}}{(\delta x)^2} \right]$$

for $i = 1, 2, 3.$ — 5M

Q. 8

S-1, P-6

$$\nabla \cdot u = 0$$

$$\nabla p - \nabla^2 u + \beta^2 u + \frac{(\kappa a)^2}{2} \rho_e \nabla \phi = 0$$

– 3M

u is scaled by $\epsilon_e E_0 \phi_0 / d_e$,

$\beta = a/l$, $l = \sqrt{K_p}$, a is the length scale

– 5M