Problem Sheet

Q1. Let $\{N(t); t \geq 0\}$ be a Poisson process with parameter λ.

(a) Find the distribution of the current life time δ_t and total life time β_t.

(b) Prove that the joint distribution of ν_t and δ_t is given by

$$P(\nu_t > x, \delta_t > y) = \begin{cases} e^{-\lambda(x+y)}, & x > 0, 0 < y < t \\ 0, & \text{if } y \geq t \end{cases}.$$

Q2. Show that the renewal function corresponding to the lifetime whose probability density function $f(x) = \lambda^2 x e^{-\lambda x},\ x \geq 0$ is $M(t) = \frac{\lambda t}{2} - \frac{1-e^{-2\lambda t}}{4}$.

Q3. Let $X_1, X_2, \ldots$ be the inter occurrence times in a renewal process. Suppose that $P(X_i = 1) = \frac{1}{3}$ and $P(X_i = 2) = \frac{2}{3}$, $i = 1, 2, \ldots$. Let N_n be the renewals upto discrete time n. Compute $P(N(1) = k), P(N(2) = k), P(N(3) = k)$.

Answers to Problem Sheet

Ans 1: **(a)** Define:

Current Life time $\delta_t = t - S_{N(t)}$

Residual Life time $\nu_t = S_{N(t)+1} - t$

Total Life time $\beta_t = \delta_t + \nu_t$

Both current and residual life time are exponentially distributed since for fixed t, $N(t) \sim P(\lambda t)$.

Hence total life time is Erlang distribution with parameter $(2, \lambda)$.

(b) $P(\nu_t > x, \delta_t > y) = P(\text{No renewals occur in the time interval } (t-y, t+x])$

Since the time to renewal is exponentially distributed, therefore the probability of renewal shall be $e^{-\lambda(x+y)}$, i.e.

$$P(\nu_t > x, \delta_t > y) = \begin{cases} e^{-\lambda(x+y)}, & x > 0, 0 < y < t \\ 0, & \text{if } y \geq t \end{cases}$$

Ans 2: We have $M(t) = E(N(t))$ renewal function.

Also $M(t) = \sum_{n=1}^{\infty} F_n(t)$ where $F_n(t)$ is cumulative distribution function of S_n.

Hence $\frac{dM(t)}{dt} = \sum_{n=1}^{\infty} f_n(t)$ where $f_n(t)$ is the probability density function of S_n.

Given that $X_n \sim G(2, \lambda)$ i.e. gamma distribution.

Hence S_n being sum n independent Gamma distributed random variables is again gamma distribution $G(2n, \lambda)$.

Hence $f_n(x) = \frac{\lambda^{2n} x^{2n-1} e^{-\lambda x}}{(2n-1)!}$.

$$
\begin{aligned}
\text{Now} \quad \frac{dM(t)}{dt} &= \sum_{n=1}^{\infty} \frac{\lambda^{2n} t^{2n-1} e^{-\lambda t}}{(2n-1)!} \\
&= \lambda e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{2n-1}}{(2n-1)!} \\
&= \lambda e^{-\lambda t} \left(\frac{e^{\lambda t} - e^{-\lambda t}}{2}\right) \\
&= \frac{\lambda}{2}(1 - e^{-2\lambda t})
\end{aligned}
$$

Hence $M(t) = \int_0^t \frac{dM(t)}{dt} dt = \int_0^t \frac{\lambda}{2}(1 - e^{-2\lambda t}) dt = \frac{\lambda t}{2} - \frac{1 - e^{-2\lambda t}}{4}$.

Ans 3: We know $P(N(t) = k) = F_k(t) - F_{k+1}(t)$ where $F_k(t) = P[S_k \leq t]$ and $S_k = X_1 + \cdots + X_k$.

Therefore, we determine the distribution of S_k:

S_k	$P(S_k = k)$
k	$\left(\frac{1}{3}\right)^k$
$k+1$	$\binom{k}{1}\left(\frac{1}{3}\right)^{k-1}\left(\frac{2}{3}\right)$
$k+2$	$\binom{k}{2}\left(\frac{1}{3}\right)^{k-2}\left(\frac{2}{3}\right)^2$
$\vdots$	$\vdots$
$2k$	$\left(\frac{2}{3}\right)^k$

Hence $S_k \sim B\left(k, \frac{2}{3}\right)$. Accordingly,

(a) $P[N(1) = k] = F_k(1) - F_{k+1}(1)$

$$F_k(1) = P[S_k \le 1] = \begin{cases} \frac{1}{3}, & k = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$F_{k+1}(1) = P[S_{k+1} \le 1] = 0$$

Hence

$$P[N(1) = k] = \begin{cases} \frac{1}{3}, & k = 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) $P[N(2) = k] = F_k(2) - F_{k+1}(2)$

$$F_k(2) = P[S_k \le 2] = P[X_1 = 1 \text{ or } (X_1 = 1, X_2 = 1) \text{ or } X_1 = 2] = \frac{10}{9}$$

$$F_{k+1}(2) = P[S_k \le 2] = \left(\frac{1}{3}\right)^2$$

Hence $P[N(2) = k] = 1$.

(c) $P[N(3) = k] = F_k(3) - F_{k+1}(3)$

$$F_k(3) = P[S_k \le 3] = \frac{43}{27}$$

$$F_{k+1}(3) = P[S_{k+1} \le 3] = \frac{16}{27}$$

Hence $P[N(3) = k] = 1$.