Problem Sheet

Q1. Show that $W^3(t)$ is an Ito process and find $d(W(t))^3$.

Q2. Let $h(t)$ be a square integrable and deterministic function. Use Ito formula to prove the identity

$$\int_0^t h(s)dW(s) \;=\; h(t)W(t) - \int_0^t h^{'}(s)W(s)ds.$$

Q3. Let $X(t) \;=\; \int_0^t e^{s-t}dW(s)$. What is the distribution of $X(t)$?

Q4. Verify $\int_0^T W(t)dW(t) \;=\; \frac{1}{2}(W(T))^2 - \frac{T}{2}$.

Q5. Prove that $\int_0^T (W(t))^2 dW(t) \;=\; \frac{(W(T))^3}{3} - \int_0^T W(t)dt$.

Answers to Problem Sheet

Ans 1: Let $f(x) \;=\; x^3$. Then

$$f^{'}(x) = \; 3x^2 \quad and \quad f^{''}(x) = \; 6x.$$

By Ito Doeblin formula version 1 we have

$$df(W(t)) \;=\; f^{'}(W(t))dW(t) + \frac{1}{2}f^{''}(W(t))dt$$

$$dW^3(t) \;=\; 3W^2(t)dW(t) + 3W(t)dt$$

Hence $W^3(t)$ satisfies above SDE and by this we can conclude that it is a Ito Process.

Ans 2: Let $f(t, x) \;=\; h(t)x$. Then

$$f_t(t, x) \;=\; h^{'}(t)x \;, \quad f_x(t, x) \;=\; x \; and \; f_{xx}(t, x) \;=\; 0.$$

By Ito Doeblin formula version 2 we have

$$f(t, W(t)) \;=\; f(0, W(0)) \;+\; \int_0^t (f_t(s, W(s)) + \frac{1}{2}f_{xx}(s, W(s)))ds \;+\; \int_0^t f_x(s, W(s))dW(s)$$

Putting all the values we get

$$\int_0^t h(s)dW(s) \;=\; h(t)W(t) - \int_0^t h^{'}(s)W(s)ds.$$

Ans 3: As $X(t)$ is Ito integral with deterministic integrand so it is normally distributed.

$$E(X(t)) \;=\; E(X(0)) \;=\; 0 \qquad \text{...(As ito integral is martingale so it has constant expectation.)}$$

$$\begin{aligned} Var(X(t)) &= E(X^2(t)) \\ &= \int_0^t e^{2(s-t)}ds \\ &= \tfrac{1-e^{-2t}}{2} \end{aligned}$$

Hence $X(t) \sim N(0, \frac{1-e^{-2t}}{2})$.

Ans 4: Let $f(x) \;=\; \frac{x^2}{2}$. Then

$$f^{'}(x) = \; x \quad and \quad f^{''}(x) = \; 1.$$

By Ito Doeblin formula version 1 we have

$$\begin{aligned} f(W(T)) &= f(W(0)) \;+\; \int_0^T f^{'}(W(t))dW(t) + \tfrac{1}{2}\int_0^T f^{''}(W(t))dt \\ \tfrac{W^2(T)}{2} &= 0 \;+\; \int_0^T W(t)dW(t) + \tfrac{1}{2}\int_0^T dt \\ \int_0^T W(t)dW(t) &= \tfrac{1}{2}(W(T))^2 - \tfrac{T}{2} \end{aligned}$$

Hence proved.

Ans 5: Let $f(x) \;=\; \frac{x^3}{3}$. Then

$$f^{'}(x) = \; x^2 \quad and \quad f^{''}(x) = \; 2x.$$

By Ito Doeblin formula version 1 we have

$$\begin{aligned} f(W(T)) &= f(W(0)) \;+\; \int_0^T f^{'}(W(t))dW(t) + \tfrac{1}{2}\int_0^T f^{''}(W(t))dt \\ \tfrac{W^2(T)}{2} &= 0 \;+\; \int_0^T W^2(t)dW(t) + \tfrac{1}{2}\int_0^T 2W(t)dt \\ \int_0^T (W(t))^2dW(t) &= \tfrac{(W(T))^3}{3} - \int_0^T W(t)dt \end{aligned}$$

Hence proved.