Problem Sheet

Q1. Let S_k be the price of risky asset at time $k = 0, 1, 2, ..., n$.

Let $S_{k+1} = \begin{cases} uS_k & \textit{with probability } p \\ dS_k & \textit{with probability } 1-p. \end{cases}$

Define a related process R_k as $R_k = \ln S_k - k(p \ln u + (1-p)\ln d)$. Prove that $\{R_k,\ k = 1, 2, ..., n\}$ is a martingale.

Q2. Let $\{X_n,\ n = 1, 2, 3, ...\}$ be a symmetric random walk and $\{F_n,\ n = 1, 2, 3...\}$ be filtration. Consider $Y_n = (-1)^n \cos(\Pi X_n), \quad n = 1, 2, 3....$ Show that $\{Y_n,\ n = 1, 2, 3...\}$ be a martingale with respect to the filtration $\{F_n,\ n = 1, 2, 3, ...\}$.

Q3. Let $X_n,\ n = 1, 2, 3...$ be a sequence of square integrable random variables. Show that if $\{X_n,\ n = 1, 2, 3, ...\}$ is a martingale with respect to the filtration $\{F_n,\ n = 1, 2, 3, ...\}$ then $\{X_n^2,\ n = 1, 2, 3, ...\}$ is a sub-martingale with respect to the same filtration.

Q4. Let $\{W(t),\ \ t \geq 0\}$ be Wiener process. Prove that $\{W^2(t) - t,\ \ t \geq 0\}$ is a martingale with respect to the natural filtration.

Q5. Let $\{W(t),\ t \geq 0\}$ be a Wiener process. Is $\{exp(\sigma W(t) - \frac{\sigma^2}{2}t),\ \ t \geq 0\}$ a martingale where σ is a positive constant?

Answers to Problem Sheet

Ans 1:

$$\begin{aligned} E(R_k/R_{k-1}, ..., R_0) &= E(\ln S_k - k(p \ln u + (1-p)\ln d)/R_{k-1}, ..., R_0) \\ &= E(\ln S_k/R_{k-1}, ..., R_0) - k(p \ln u + (1-p)\ln d) \\ &= E(\ln S_k/S_{k-1}, ..., S_0) - k(p \ln u + (1-p)\ln d) \\ &= \ln S_{k-1} + p \ln u + (1-p)\ln d - k(p \ln u + (1-p)\ln d) \\ &= \ln S_{k-1} - (k-1)(p \ln u + (1-p)\ln d) \\ &= R_{k-1} \qquad \forall\ \ k = 1, 2, 3..., n \end{aligned}$$

Hence $\{R_k,\ k = 1, 2, ..., n\}$ is a martingale.

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Ans 2: We know for symmetric random walk $E(X_n) = 0$.

It can be checked that $E(Y_n) < \infty \quad \forall \; n$.

$$\begin{aligned}
E(Y_{n+1}/F_n) &= E((-1)^{n+1}\cos(\Pi X_{n+1})/F_n) \\
&= E((-1)^{n+1}\cos(\Pi(X_{n+1} - X_n + X_n))/F_n) \\
&= E((-1)^{n+1}\cos(\Pi(Z_{n+1} + X_n))/F_n) \\
&= (-1)^{n+1}E(\cos(\Pi Z_{n+1})\cos(\Pi X_n)/F_n) - (-1)^{n+1}E(\sin(\Pi Z_{n+1})\sin(\Pi X_n)/F_n) \\
&= (-1)^{n+1}\cos(\Pi X_n)E(\cos(\Pi Z_{n+1})/F_n) - (-1)^{n+1}\sin(\Pi X_n)E(\sin(\Pi Z_{n+1})/F_n) \\
&= (-1)^{n+1}\cos(\Pi X_n)E(\cos(\Pi Z_{n+1})) - (-1)^{n+1}\sin(\Pi X_n)E(\sin(\Pi Z_{n+1})) \\
&= (-1)^{n+1}\cos(\Pi X_n)(-1) - (-1)^{n+1}\sin(\Pi X_n)(0) \\
&= (-1)^{n}\cos(\Pi X_n) \\
&= Y_n
\end{aligned}$$

Hence $\{Y_n, \; n = 1, 2, 3...\}$ is a martingale.

Ans 3: Let $\Phi(x) \; = \; x^2$. It is a convex function. So by Jensen's inequality

$$E(\Phi(X_{n+1})/F_n) \; \geq \; \Phi(E(X_{n+1}/F_n))$$

$$E(X_{n+1}^2/F_n) \; \geq \; \Phi(X_n) \; = \; X_n^2.$$

$\Rightarrow \;\; X_n^2$ is a sub-martingale.

Ans 4: Let $\{F(t); \; t \geq 0\}$ be natural filtration.

Consider $s < t$

$$\begin{aligned}
E(W^2(t) - t/F(s)) &= E((W(t) - W(s) + W(s))^2 - t/F(s)) \\
&= E((W(t) - W(s))^2 + W^2(s) + 2(W(t) - W(s))W(s) - t/F(s)) \\
&= E[(W(t) - W(s))^2] + W^2(s) + 2W(s)E[(W(t) - W(s))] - t \\
&= t - s + W^2(s) + 2W(s) * 0 - t \\
&= W^2(s) - s
\end{aligned}$$

Thus $\{W^2(t) - t, \;\; t \geq 0\}$ is a martingale with respect to the natural filtration.

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Ans 5: Let $\{F(t);\ t \geq 0\}$ be natural filtration.

Consider $s < t$

$$\begin{aligned}
E(exp(\sigma W(t) - \tfrac{\sigma^2}{2}t)/F(s)) &= exp(-\tfrac{\sigma^2}{2}t))E(exp(\sigma W(t))/F(s)) \\
&= exp(-\tfrac{\sigma^2}{2}t))E(exp(\sigma(W(t) - W(s) + W(s)))/F(s)) \\
&= exp(-\tfrac{\sigma^2}{2}t))exp(\sigma W(s))E(exp(\sigma(W(t) - W(s)))/F(s)) \\
&= exp(-\tfrac{\sigma^2}{2}t))exp(\sigma W(s))E(exp(\sigma(W(t) - W(s)))) \\
&= exp(-\tfrac{\sigma^2}{2}t))exp(\sigma W(s))exp(\sigma * 0 + \tfrac{1}{2}\sigma^2(t-s)) \\
&= exp(\sigma W(s) - \tfrac{\sigma^2}{2}s)
\end{aligned}$$

Thus $\{exp(\sigma W(t) - \frac{\sigma^2}{2}t),\ \ t \geq 0\}$ is a martingale with respect to the natural filtration.

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