Problem Sheet

1. Two communication satellites are placed in an orbit. The lifetime of a satellite is exponential with mean $1/u$. If one fails, its replacement to sent up. The time necessary to prepare and send up a replacement is exponential with mean $\frac{1}{\lambda}$. Let X_t =number of satellites in the orbit at time t. Assume $\{X_t : t \geq 0\}$ to be a Markov process with stat space $\{0, 1, 2\}$. Give the infinitesimal generator matrix and Kolmogrov equations for the process.

2. Suppose that a company has 4 operators serving a single telephone number. Anybody who calls while all four operators are busy will receive a busy signal. Let X_t = number of busy operation at time t. Assume arrival and individual service process to be poisson with rates λ and μ. Determine the infinitesimal generator matrix and the forward Kolmogrov equations for the stated Markov process.

3. A birth death process is called a pure death process if $\lambda_i = 0\ \forall i$(i.e. no arrival takes place). Suppose $\mu_i = i\mu, i = 1, 2, 3, \ldots$ and initially $X_0 = n$, then show that $X_t \sim B(n, p)$ with $p = e^{-\mu t}$.

4. Suppose that a man operates a small auto collision shop rate λ. The man first bumps a car, then paints and thereafter starts on the next car. The length of time to bump a car is exponential with mean $\frac{1}{\mu_1}$ and length of time to paint a car is exponential with mean $\frac{1}{\mu_2}$. Model the process as a Markov process $\{X_t : t \geq 0\}$ where $X_t = (i, j)$ if there are $i > 0$ cars in the shop and the car being in stage j where $j = 0$ represents bumping stage and $j = i$ as the painting stage. if the shop is empty $X_t = (0, 0)$. Determine the infinitesimal generator matrix.

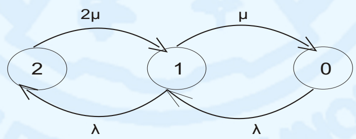
5. A rural telephone switch has C circuits available to carry C calls. A new call is blocked if all circuits are busy. Suppose calls have duration which is exponentially distributed with mean $\frac{1}{\mu}$ and inter-arrival time of calls is exponential with mean $\frac{1}{\lambda}$. Assume calls arrive independently and are served independently. model this process as a birth-death process and write the forward kolmogrov equations for the process. Also find the probability that a call is blocked when the system is in steady state.

6. The arrival of a large number of jobs at a server firms with Poisson process with rate 2/hr. The service times of such jobs is exponentially distributed with mean 20 min. Only four jobs can be accommodated in the system at a time. Assuming that the fraction of computing power utilized by smaller jobs is negligible, determine the probability that a large job will be turned away because of lack of storage space. Also find the mean number of large jobs in the system at steady state.

7. A digital camera needs three batteries to run. You buy a pack of six batteries and install three of them into the camera. Whenever a battery is drained, you immediately replace the drained battery with a new one available in stock. Assume that each battery lasts for an amount of time that is exponentially distributed with mean $1/\mu$, independent of other batteries. Eventually the camera stops running when only two batteries are left. Let $X(t)$ denote the number of batteries not drained at time t. Find the expected time that the camera will be able to run with the purchased pack of batteries.

Answers to Problem Sheet

1. The state transition diagram is given by:



Note that there are 2 satellites in the orbit. Also, failure rate of anyone= μ and repair rate = λ.

$\therefore$ infinitesimal generator matrix is given by:

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & 2\mu & -2\mu \end{bmatrix}$$

Further, the Kolmogrov equations are:

<u>Forward:</u> $P'(t) = P(t)Q$

i.e. $$\begin{bmatrix} p'_{00}(t) & p'_{01}(t) & p'_{02}(t) \\ p'_{10}(t) & p'_{11}(t) & p'_{12}(t) \\ p'_{20}(t) & p'_{21}(t) & p'_{22}(t) \end{bmatrix} = \begin{bmatrix} p_{00}(t) & p_{01}(t) & p_{02}(t) \\ p_{10}(t) & p_{11}(t) & p_{12}(t) \\ p_{20}(t) & p_{21}(t) & p_{22}(t) \end{bmatrix} Q$$

$\therefore p'_{00}(t) = -\lambda p_{00}(t) + \mu p_{01}(t)$

$p'_{01}(t) = \lambda p_{00}(t) - (\lambda + \mu)p_{01}(t) + \lambda p_{02}(t)$

$p'_{02}(t) = \lambda p_{01}(t) - 2\mu p_{02}(t)$

$p'_{10}(t) = -\lambda p_{10}(t) + \mu p_{11}(t)$

$p'_{11}(t) = \lambda p_{10}(t) - (\lambda + \mu)p_{11}(t) + 2\mu p_{12}(t)$

$p'_{12}(t) = \lambda p_{11}(t) - 2\mu p_{12}(t)$

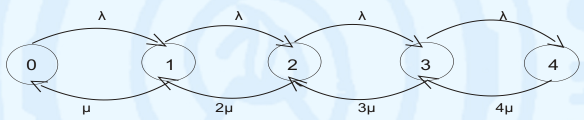
$p'_{20}(t) = -\lambda p_{20}(t) + \mu p_{21}(t)$

$p'_{21}(t) = \lambda p_{20}(t) - (\lambda + \mu)p_{21}(t) + 2\mu p_{22}(t)$

$p'_{20}(t) = \lambda p_{21}(t) - 2\mu p_{22}(t)$

Backward: $P'(t) = QP(t)$

2. The state space is $\{0, 1, 2, 3, 4\}$ and state transition diagram is given by:



The infinitesimal generator matrix is given by:

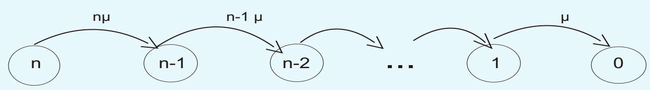
$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda & 0 \\ 0 & 0 & 3\mu & -(\lambda+3\mu) & \lambda \\ 0 & 0 & 0 & 4\mu & -4\mu \end{bmatrix}$$

The forward Kolmogrov equations are given by:

$P'(t) = P(t)Q$, i.e.

$$\begin{bmatrix} p'_{00}(t) & p'_{01}(t) & p'_{02}(t) & p'_{03}(t) & p'_{04}(t) \\ p'_{10}(t) & p'_{11}(t) & p'_{12}(t) & p'_{13}(t) & p'_{14}(t) \\ p'_{20}(t) & p'_{21}(t) & p'_{22}(t) & p'_{23}(t) & p'_{24}(t) \\ p'_{20}(t) & p'_{21}(t) & p'_{22}(t) & p'_{23}(t) & p'_{24}(t) \\ p'_{30}(t) & p'_{31}(t) & p'_{32}(t) & p'_{33}(t) & p'_{34}(t) \\ p'_{40}(t) & p'_{41}(t) & p'_{42}(t) & p'_{43}(t) & p'_{44}(t) \end{bmatrix} = \begin{bmatrix} p_{00}(t) & p_{01}(t) & p_{02}(t) & p_{03}(t) & p_{04}(t) \\ p_{10}(t) & p_{11}(t) & p_{12}(t) & p_{13}(t) & p_{14}(t) \\ p_{20}(t) & p_{21}(t) & p_{22}(t) & p_{23}(t) & p_{24}(t) \\ p_{20}(t) & p_{21}(t) & p_{22}(t) & p_{23}(t) & p_{24}(t) \\ p_{30}(t) & p_{31}(t) & p_{32}(t) & p_{33}(t) & p_{34}(t) \\ p_{40}(t) & p_{41}(t) & p_{42}(t) & p_{43}(t) & p_{44}(t) \end{bmatrix} Q$$

3. The state transition diagram is:



Initial state is $X_0 = n \Rightarrow \Pi_n(0) = 1$ and $\Pi_k(0) = 0 \ \forall \ 0 \leq k \leq n-1$.

The differential equations are:

$\Pi_n'(t) = -n\mu\Pi_n(t)$

$\Pi_k'(t) = -k\mu\Pi_k(t) + (k+1)\mu\Pi_{k+1}(t)$ for $1 \leq k \leq n-1$

$\Pi_o'(t) = \mu\Pi_1(t)$.

Taking laplace transform on both sides and using initial condition and then again taking inverse laplace transform we get,

$\Pi_k(t) = {}^nC_k(e^{-\mu t})^k(1-e^{-\mu t})^{n-k}, \ 0 \leq k \leq n, \ t \geq 0.$

For a fixed t, this is the binomial probability mass function with parameters n and $p = e^{-\mu t}$.

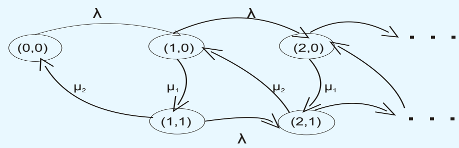
4. Given arrival rate= λ

Service rate for bumping= μ_1

Service rate for painting = μ_2

$X_{ij} = (i,j), i = 0,1,2,\ldots \ \& \ , \ j = \left\{ \begin{array}{ccc} 0 & : & bumping \\ 1 & : & painting \end{array} \right\}$

The state transition diagram is:

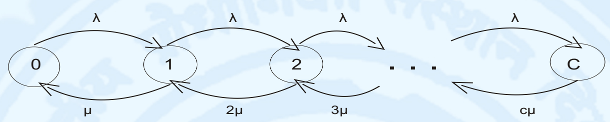


The infinitesimal generator matrix is:

00 10 11 20 21 ...

$$\begin{array}{c} 00 \\ 10 \\ 11 \\ 20 \\ 21 \\ \vdots \end{array} \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \dots & \dots \\ 0 & -(\lambda+\mu_1) & \mu_1 & \lambda & 0 & \dots \\ \mu_2 & 0 & -(\lambda+\mu_2) & 0 & \lambda & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

5. The state transition diagram for the given problem is:



The system can be modelled as an M/M/c/c queuing system

The kolmogrov equations are given by:

$p_0'(t) = -\lambda p_0(t) + \mu p_1(t)$

$p_k'(t) = \lambda p_{k-1}(t) - (\lambda + \mu) p_k(t) + 2\mu p_{k+1}(t),\ k = 1, \dots, c-1$

$p_c'(t) = -c\mu p_c(t) + \lambda p_{c-1}(t)$

The steady state probability of the system is:

$p_i = \frac{1}{i!}(\frac{\lambda}{\mu})^i p_0,\ \ i = 1, \dots, c$

$p_0 = \left[\sum_{i=0}^{c} \frac{1}{i!}(\frac{\lambda}{\mu})^i\right]^{-1}$

The probability that a call is blocked in steady state is:

$p_c = \frac{\frac{1}{c!}(\frac{\lambda}{\mu})^c}{\sum_{i=0}^{c} \frac{1}{i!}(\frac{\lambda}{\mu})^i}$

6. The system can be modeled as $M/M/4/4$ queueing system with arrival rate $\lambda = 2$hr and service rate $\mu = 3/hr$. Then for $p_i = P$ (i jobs in the system) we get:

$$p_i = \frac{1}{i!}(\frac{\lambda}{\mu})^i p_0, \quad i = 1, 2, 3, 4$$

where $p_0 = [\sum_{i=0}^{4} \frac{1}{i!}(\frac{\lambda}{\mu})^i]^{-1} = 0.5137$

Hence

P (job is turned away because of lack of storage space)

$=P$ (system is full)

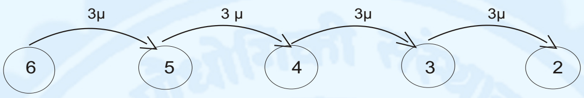
$=p_4$=.00423

Mean no. of jobs in the system at steady state

=.3425+2×.1142+3×.02537+4 × .00423

=0.66393

7. The state space of the given process is: $\{6, 5, 4, 3, 2\}$. The state transition diagram is:



The rate is 3μ since anyone of the three installed batteries may fail.

The camera works in each of the above states except state 2.

The average time that the camera works in each stage is $\frac{1}{3\mu}$. ∴ The expected time that the camera runs is $\frac{4}{3\mu}$.