Problem Sheet

Q1. Let X_n be a Markov chain with state space $\{0, 1, 2\}$ and initial probability vector $p(0) = (1/4, 1/2, 1/4)$ and one-step transition matrix: $\begin{pmatrix} 1/4 & 3/4 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/4 & 3/4 \end{pmatrix}$

(a) Compute $P(X_0 = 0, X_1 = 1, X_2 = 1),\ P(X_2 = 1)$

(b) Compute $P(X_1 = 1, X_2 = 1 \mid X_0 = 0)$

(c) Compute $P(X_4 = 1 \mid X_2 = 2),\ P(X_7 = 0 \mid X_5 = 0)$.

Q2. Consider a communication system which transmits the two digits 0 and 1 through several stages. Let X_0 be the digit transmitted initially (leaving)0^{th} stage and $X_n,\ n \geq 1$ be the digit leaving the n^{th} stage. At each stage there is a constant probability q that the digit which enters is transmitted unchanged and the probability p otherwise with $p + q = 1$. Show that $\{X_n : n \geq 0\}$ is a Markov chain. Find one-step transition probability matrix P and compute P^m; $lim_{m\to\infty} P^m$; $P(X_0 = 0 \mid X_m = 0)$ and $P(X_m = 0)$.

Q3. A factory has two machines and one crew. Assume that the probability of any one machine breaking down on a given day is α. Assume that if the repair crew is working on a machine, the probability that they will complete the repair in one day is β. For simplicity, ignore the probability of a repair completion or a breakdown taking place except at the end of the day. Let X_n denote the number of machines in operation at the end of the n^{th} day. Assume that the behavior of X_n can be Markov chain.

(a) Find one step transition matrix for the chain.

(b) If the system starts out with both machines operation then what will be the probability that both will be in operation two days later?

Q4. Show that if a markov chain is irreducible and $P_{ii} > 0$ for some i, then the chain is aperiodic.

Q5. Consider a DTMC with states 0,1,2,3,4. Suppose $p_{0,4} = 1$ and suppose that when the chain is in the state i, $i > 0$, the next state is equally likely to be any of the states 0,1,2,...,i-1.

(a) Discuss the nature of states of this Markov chain.

(b) Discuss whether there exits a limiting distribution and find one if it exists.

Answers to Problem Sheet

Ans 1: Given: Markov chain X_n and state space $\{0, 1, 2\}$.

$p(0) = (1/4, 1/2, 1/4)$ and transition probability matrix

$$\begin{pmatrix} 1/4 & 3/4 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/4 & 3/4 \end{pmatrix}$$

(a)(i) $P(X_0 = 0, X_1 = 1, X_2 = 1)$

$= P(X_2 = 1 \mid X_1 = 1)P(X_1 \mid X_0 = 0)P(X_0 = 0) = 1/3.1/3.1/4 = 1/36$

(ii) $P(X_2 = 1)$. Consider:

$$P^2 = \begin{pmatrix} 1/4 & 3/4 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/4 & 3/4 \end{pmatrix} \begin{pmatrix} 1/4 & 3/4 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/4 & 3/4 \end{pmatrix} = \begin{pmatrix} 5/16 & 7/16 & 1/4 \\ 7/36 & 16/36 & 13/36 \\ 1/12 & 13/48 & 31/48 \end{pmatrix}$$

$$\begin{aligned} \therefore\ P(X_2 = 1) &= P(X_2 = 1 \mid X_0 = 0)P(X_0 = 0) + P(X_2 = 1 \mid X_0 = 1)P(X_0 = 1) \\ &+ P(X_2 = 1 \mid X_0 = 2)P(X_0 = 2) \\ &= 7/16 \times 1/4 + 16/36 \times 1/2 + 13/48 \times 1/4 = 0.3993 \end{aligned}$$

(b) $P(X_1 = 1, X_2 = 1 \mid X_0 = 0) = \frac{P(X_0, X_1=1, X_2=1)}{P(X_0=0)} = \frac{1}{9}$

(c) $P(X_4 = 1 \mid X_2 = 2) = 13/48$

$P(X_7 = 0 \mid X_5 = 0) = 5/16$

This is because going from 2 to 4 or 5 to 7 we are moving by 2 steps only. Hence we make use of 2-step transition matrix P^2.

Ans 2: $X_n =$ digit transmitted at the n^{th} stage.

Let P (digit transmitted is unchanged) $= P$

P (digit transmitted is unchanged) $= q$

Then for P_{ij} denoting the probability of going from state i to j, we get the transition probability matrix as: $P = \begin{pmatrix} p & q \\ q & p \end{pmatrix}$

Further, P^m can be computed inductively as follows:

$$P^2 = \begin{pmatrix} p & q \\ q & p \end{pmatrix} \begin{pmatrix} p & q \\ q & p \end{pmatrix} = \begin{pmatrix} p^2+q^2 & 2pq \\ 2pq & p^2+q^2 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} p^2+q^2 & 2pq \\ 2pq & p^2+q^2 \end{pmatrix} \begin{pmatrix} p & q \\ q & p \end{pmatrix} = \begin{pmatrix} p^3+3pq^2 & q^3+3p^2q \\ 3p^2q+q^3 & p^3+3pq^2 \end{pmatrix}$$

$$P^4 = \begin{pmatrix} p^2+q^2 & 2pq \\ 2pq & p^2+q^2 \end{pmatrix} \begin{pmatrix} p^2+q^2 & 2pq \\ 2pq & p^2+q^2 \end{pmatrix} = \begin{pmatrix} p^4+q^4+6p^2q^2 & (p^2+q^2)(pq+3pq) \\ (p^2+q^2)(pq+3pq) & p^4+q^4+6p^2q^2 \end{pmatrix}$$

Hence inductively:

$$P^m = \begin{pmatrix} 1/2+1/2(q-p)^m & 1/2-1/2(q-p)^m \\ 1/2-1/2(q-p)^m & 1/2+1/2(q-p)^m \end{pmatrix}$$

Let $P_{ij}^m = 1/2 \ \forall \ i,j = 0,1.$

Hence $Lt_{m\to\infty} P^{(m)} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

$$\begin{aligned} P(X_0 = 0 \mid X_m = 0) &= \tfrac{P(X_m=0 \mid X_0=0)P(X_0=0)}{P(X_m=0)} \\ &= 1/2[1/2 + 1/2(q-p)^m] \\ P(X_m = 0) &= P(X_m = 0 \mid X_0 = 0)P(X_0 = 0) + P(X_m = 0 \mid X_0 = 1)P(X_0 = 1) \\ &= 1/2 \end{aligned}$$

Ans 3: Let $X_n =$ no. of machines in operation at the end of n^{th} day

P (1 machine breaks down)$= \alpha$

P (1 machine is repaired)$= \beta$

(a) Then 1-step transition matrix is given by:

$p_{00} = 1-\beta,\quad p_{01} = \beta,\quad p_{02} = 0$

$p_{10} = \alpha(1-\beta),\quad p_{11} = \alpha\beta + (1-\alpha)(1-\beta),\quad p_{02} = (1-\alpha)\beta$

$p_{20} = \alpha^2,\quad p_{21} = 2\alpha(1-\alpha),\quad p_{22} = (1-\alpha)^2$

$$\text{i.e. } P = \begin{pmatrix} 1-\beta & \beta & 0 \\ \alpha(1-\beta) & \alpha\beta+(1-\alpha)(1-\beta) & (1-\alpha)\beta \\ \alpha^2 & 2\alpha(1-\alpha) & (1-\alpha)^2 \end{pmatrix}$$

(b) To find $P(X_2 \mid X_0 = 2)$

$$\begin{aligned} P(X_2 = 2 \mid X_0 = 2) &= p_{22}^{(2)} \\ &= \alpha\beta(1-\beta) + [\alpha\beta + (1-\alpha)(1-\beta)]^2 + 2\alpha\beta(1-\alpha)^2 \end{aligned}$$

Ans 4: Since $P_{ii} > 0 \;\; \Rightarrow \;\; d_i \; = \; 1.$

Also as Markov chain is irreducible so all the states have same period that is 1.

Hence Markov chain is aperiodic

Ans 5: (a) As all the states can be reached from one another, so all are recurrent.

(b) We know if chain is aperiodic then limiting distribution exists.

$p_{00}^{(2)} = 1 * P_4 \; > 0, \;\; p_{00}^{(3)} = 1 * P_4 * p_3 \; > 0$

$So\ that\ \ d_0 \; = \; G.C.D.\ \{2, 3, ...\} \; = \; 1. \qquad p_{11}^{(3)} \; = \; p_1 * 1 * p_4 \; > 0, \quad p_{11}^{(4)} \; = \; p_1 * 1 * p_4 * p_3 \; > 0.$

$So\ that\ \ d_1 \; = \; G.C.D.\{3, 4, ...\} \; = \; 1$

Similarly we can calculate $d_2 = d_3 = d_4 = 1$.

As $d_i \; = \; 1 \quad \forall \;\; i$. So chain is aperiodic. Hence limiting distribution exists.

$$\text{Transition probability matrix is P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ p_1 & 0 & 0 & 0 & 0 \\ p_2 & p_2 & 0 & 0 & 0 \\ p_3 & p_3 & p_3 & 0 & 0 \\ p_4 & p_4 & p_4 & p_4 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}$$

Limiting distribution can be found by solving system of equations $V = VP \quad and \quad \sum_{i=0}^{4} v_i = 1$.

By solving we get

$$V = [v_0 \ v_1 \ v_2 \ v_3 \ v_4] = [\tfrac{12}{37} \ \tfrac{6}{37} \ \tfrac{4}{37} \ \tfrac{3}{37} \ \tfrac{12}{37}].$$