Problem Sheet

1. Show that an i.i.d sequence of continuous random variable with common probability density function f is strictly stationary.

2. Find(under certain conditions) whether the stochastic process $\{X(t), t \in T\}$ with probability distribution given by:
$$P(X(t) = n) = \left\{ \begin{array}{ll} \frac{(at)^{n-1}}{(1+at)^{n+1}}, & n = 1, 2, \dots \\ \frac{at}{1+at}, & n = 0 \end{array} \right\}$$
is stationary.

3. Let $X(t) = A_0 + A_1 t + A_2 t^2$ where $A_i' s$ are uncorrelated random variables with mean 0 and variance 1. Find the mean function and covariance function of $X(t)$.

4. Let $Y_n = a_0 X_n + a_1 X_{n-1}$, $n = 1, 2, \dots$ where a_0, a_1 are constants and $X_0, X_1, \dots,$ are i.i.d. random variables with mean 0 and variance σ^2.

   (a) Is $\{Y_n, n \geq 1\}$ covariance stationary?

5. Consider autoregressive process of order 1, i.e.
$$X_t = c + \phi X_{t-1} + \varepsilon_t$$
where ε_t is white noise with mean 0 and variance ${\sigma_\varepsilon}^2$, c is a constant.Assume that the mean of the random variable X_t is identical for all values of t, denoted by μ. Show that the process is wide sense stationary for $|\phi| < 1$.

6. Let $\{N(t),\ t \geq 0\}$ be a Poisson Process. Prove or disprove that $\{X(t) = N(t+L) - N(t),\ t \geq 0\}$, where L is a positive constant, is covariance or wide-sense stationary.

7. Let Z_1 *and* Z_2 be two independent normal random variables with mean 0 and variance σ^2. Define $X(t) =\ Z_1 cos(\lambda t) + Z_2 sin(\lambda t)$. Then show that $\{X(t),\ t \in T\}$ is a second order stationary process.

Answers to Problem Sheet

Ans 1. Let $X_1, X_2, \ldots,$ be an i.i.d. sequence of continuous random variables.

Let n be any positive integer.

Let $m \in Z$ such that $n + m > 0$.

Then $P(X_{1+m}, X_{2+m}, \ldots, X_{n+m}) \in B$ and its distribution is:

$$\int\int \ldots \int_B f(x_{1+m}) f(x_{2+m}) \ldots f(x_{n+m}) dx_{1+m} dx_{2+m} \ldots dx_{n+m}$$

Since $X_i' s$ are i.i.d. random variables and $x_{1+m}, x_{2+m} \ldots x_{n+m}$ are just dummy variables of integration, we may replace them by $x_1, x_2, \ldots, x_n$.

Hence above integral is equal to

$$\int \ldots \int_B f(x_1) f(x_2) \ldots f(x_n) dx_1 dx_2 \ldots dx_n$$

which is independent of m and hence the process is strictly stationary.

Ans 2. Given $P[X(t) = n] = \left\{ \begin{array}{cc} \frac{(at)^{n-1}}{(1+at)^{n+1}}, & n = 1, 2, \ldots \\ \frac{at}{1+at}, & n = 0 \end{array} \right\}$

(i) $E[X(t)] = \sum_0^\infty n P(X(t) = n) = \sum_1^\infty \frac{n(at)^{n-1}}{(1+at)^{n+1}}$

$= \frac{1}{(1+at)^2} \sum_1^\infty n \left[\frac{at}{1+at}\right]^{n-1} = \frac{1}{(1+at)^2} \cdot (1+at)^2 = 1$

(ii) $E[X^2(t)] = \sum n^2 \frac{(at)^{n-1}}{(1+at)^{n+1}} = \frac{1}{(1+at)^2} \sum_1^\infty n^2 \left(\frac{at}{1+at}\right)^{n-1}$

$= \frac{1}{(1+at)^2} \cdot (1 + 2at)$

Ans 3. Let $X(t) = A_0 + A_1 t + A_2 t^2$ where

$E(A_i) = 0 \ \forall \ i$, $\text{Var}(A_i) = 1 \ \forall \ i$ and $\text{Cov}(A_i, A_j) = 0 \ \forall \ i \neq j$.

(a) Mean function of $X(t)$:

$E[X(t)] = E[A_0 + A_1 t + A_2 t^2] = E[A_0] + tE[A_1] + t^2 E[A_2] = 0$

(b) Covariance function of $X(t)$:

$$\begin{aligned} \text{Cov}(X(t_1), X(t_2)) &= E[X(t_1)X(t_2)] - E[X(t_1)]E[X(t_2)] \\ &= E[X(t_1)X(t_2)] \\ &= E[(A_0 + A_1 t_1 + A_2 t_1^2)(A_0 + A_1 t_2 + A_2 t_2^2)] \\ &= E[A_0^2 + A_0 A_1 t_2 + A_0 A_2 t_2^2 + A_1 A_0 t_1 + A_1^2 t_1 t_2 + A_1 t_1 t_2^2 + A_0 A_2 t_1^2 + A_1 A_2 t_1^2 t_2 \end{aligned}$$

$+ A_2^2 t_1^2 t_2^2]$

Now, as $\text{Cor}(A_i, A_j) = 0 \ \forall \ i \neq j$, therefore:

$E[A_iA_j] - E[A_i]E[A_j] = 0 \ \forall \ i \neq j, \Rightarrow E[A_iA_j] = E[A_i]E[A_j]$ and $\text{Var}A_i = E[A_i^2]$

Hence

$$\begin{aligned}\text{Cov}(X(t_1), X(t_2)) &= E[A_0^2]+t_2E[A_0]E[A_1]+t_2^2E[A_0]E[A_2]+t_1E[A_1]E[A_0]+t_1t_2E[A_1^2]+t_1t_2^2E[A_1]E[A_2]+\\ &\quad t_1^2E[A_0]E[A_2] + t_1^2t_2E[A_1]E[A_2] + t_1^2t_2^2E[A_2^2]\\ &= 1 + t_1t_2 + t_1^2t_2^2 \ \ (\because E[A_i] = 0 \ \forall i).\end{aligned}$$

Ans 4. $Y_n = a_0X_n + a_1X_{n-1}$, $n = 1, 2, \ldots$ where a_i's are constants and $X_0, X_1, \ldots,$ are i.i.d's random variables with $\text{E}(X_i) = 0$ and $\text{Var}X_i = \sigma^2$.

(a) Is Y_n covariance stationary:

(i)$E[Y_n] = E[a_0X_n + a_1X_{n-1}] = 0$

$$\begin{aligned}\text{(ii)}E[Y_n^2] &= E[(a_0X_n + a_1X_{n-1})^2]\\ &= E[a_0^2X_n^2 + a_1^2X_{n-1}^2 + 2a_0a_1X_nX_{n-1}]\\ &= a_0^2\sigma^2 + a_1^2\sigma^2 + 2a_0a_1E(X_nX_{n-1})\\ &= a_0^2\sigma^2 + a_1^2\sigma^2 + a_0a_1(E(X_n)E(X_{n-1})) \ \ (\because \text{ they are i.i.d})\\ &= a_0^2\sigma^2 + a_1^2\sigma^2(\because \text{E}(X_i) = 0)\end{aligned}$$

$$\begin{aligned}\text{(iii)}\text{Cov}(Y_n, Y_m) &= \text{Cov}(a_0X_n + a_1X_{n-1}, a_0X_m + a_1X_{m-1})\\ &= E[(a_0X_n + a_1X_{n-1})(a_0X_m + a_1X_{m-1})](\because E(Y_n) = E(Y_m) = 0)\\ &= E[a_0^2X_nX_m + a_0a_1X_nX_{m-1} + a_1a_0X_mX_{n-1} + a_1^2X_{m-1}X_{n-1}]\\ &= \begin{cases} a_0^2\sigma^2 + a_1^2\sigma^2, & \text{n=m;}\\ a_0a_1\sigma^2, & \text{n=m-1;}\\ a_0a_1\sigma^2, & \text{n=m+1;}\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

which is a function of $n - m$.

Hence Y_n is covariance stationary.

Ans 5. **(i)** First calculating expectation

$$E(X_t) = E(c + \phi X_{t-1} + \varepsilon_t)$$
$$\mu = c + \phi\mu + 0$$
$$\Rightarrow \mu = \frac{c}{1-\phi}$$

which is independent of t.

$$\text{Var}(X_t) = \sigma_X^2 \text{ and} \quad (1)$$

(ii) $$\text{Var}(X_t) = Var[c + \phi X_{t-1} + \varepsilon_t]$$
$$= \phi^2 \text{Var}(X_{t-1}) + {\sigma_\epsilon}^2 \quad (2)$$

Since $\{X_t : t \in T\}$ are identical, $\therefore$ $\text{Var}(X_t) = \text{Var}(X_{t-1})$

Equating (1) and (2):

$$\sigma_X^2 = \phi^2 \sigma_X^2 + {\sigma_\varepsilon}^2$$
$$\sigma_X^2 = \frac{{\sigma_\varepsilon}^2}{1-\phi^2} \Rightarrow \text{Var}(X_t) = \frac{{\sigma_\varepsilon}^2}{1-\phi^2}$$

which exists and is finite for $|\phi| < 1$.

(iii) Since X_t's are identical

$$E(X_{t_1} X_{t_2}) = \mu^2 \text{ and}$$
$$\text{Cov } (X_{t_1}, X_{t_2}) = 0$$

which are functions of $|t_1 - t_2|$.

Hence the process is wide sense stationary.

Ans 6. We have $X(t) = N(t+L) - N(t) \sim P(\lambda(t+L-t)) = P(\lambda L)$

(a) $E(X(t)) = \lambda L$ which is independent of t.

(b) $E(X^2(t)) = \lambda L + (\lambda L)^2 < \infty \quad \forall t.$

(c) Let $s < t$.

$$cov(X(t), X(s)) = E(X(t)X(s)) - E(X(t))E(X(s))$$
$$= E((X(t) - X(s) + X(s))X(s)) - (\lambda L)^2$$
$$= E(X(t) - X(s))E(X(s)) + E(X^2(s)) - (\lambda L)^2$$
$$= 0 * E(X(s)) + \lambda L$$
$$= \lambda L$$

which is constant function. So we can consider it as a function of $t - s$.

From (a),(b) and (c) $\{X(t), \quad t \geq 0\}$ is covariance stationary.

Ans 7. (a) $E(X(t)) \;=\; E(Z_1)cos(\lambda t) + E(Z_2)sin(\lambda t)$

$= \; 0$ which is independent of t.

(b) $E(X^2(t)) \;=\; cos^2(\lambda t)E(Z_1^2) + sin^2(\lambda t)E(Z_2^2) + 2cos(\lambda t)sin(\lambda t)E(Z_1)E(Z_2)$

$= \; cos^2(\lambda t)\sigma^2 + sin^2(\lambda t)\sigma^2 + 2cos(\lambda t)sin(\lambda t) * 0$

$= \; \sigma^2 \; < \qquad \forall \;\; t.$

From (a),(b) $\{X(t), \;\; t \geq 0\}$ is second order stationary.