Problem Sheet

1. A flight from Los Vegas to New Delhi had stopover at New York and London. In this process the luggage had been transferred three times. The probability of the 1st transfer being delayed is $\frac{6}{10}$. The probability that the second transfer was delayed, due to delay in the first transfer is $\frac{2}{10}$. The delay in the third transfer, due to delay in the first and second transfer, has a probability of $\frac{1}{10}$. What is the probability that there is delay in all the three transfers?

2. If $X \sim N(0, \sigma^2)$, then calculate $\mathrm{E}(X^4)$.

3. Suppose that the life length (in hours) of a certain radio tube is a continuous random variable X with pdf: $f_X(x) = \frac{100}{x^2}$ if $x > 100$ and 0 elsewhere. Then:

   (a) What is the probability that the tube will last less than zero hours if it is known that the tube is still functioning after 150 hours of service.

   (b) What is the probability that if 3 such tubes are installed in set, exactly one will have to be replaced after 150 hours of service.

   (c) What is the minimum number of tubes that may be into a set so that there is a probability of almost 0.25 that after 150 hours of service, none of them are functioning.

4. The waiting time, denoted by X, in hours, between successive speeders spotted by a radar unit is a continuous random variable with c.d.f:

$$F_X(x) = \begin{cases} 0, & x < 0; \\ 1 - e^{-4x}, & x \geq 0. \end{cases}$$

   (a) Find the probability density function of the random variable X.

   (b) Find the probability of waiting less than 12 minutes between successive speeders.

   (c) Find the probability that waiting time is less than 30 minutes given that no speeder is spotted till 15 minutes.

5. Let $X \sim Exp(\lambda)$. Prove that the random variable X has "no memory", that is :$\mathrm{P}(X > r + s|x > r) = \mathrm{P}(X > s)$ for any two positive real numbers r and s.

Answers to Problem Sheet

Ans 1. Let the event of delay in the ith transfer be denoted by A_i. Then we have the following information.

$P(A_1) = \frac{6}{10}$, $P(A_2|A_1) = \frac{2}{10}$, $P(A_3|A_1A_2) = \frac{1}{10}$.

To find: P$(A_1A_2A_3)$.

By the law of Multiplication, we have:

P$(A_1A_2A_3)$=$P(A_3|A_1A_2)P(A_2|A_1)P(A_1)$.

$$= \frac{6}{10}\cdot\frac{2}{10}\cdot\frac{1}{10}$$

$$= \frac{12}{1000}$$

Ans 2. E$(X^4) = \int_{-\infty}^{\infty} \frac{x^4}{\sqrt{2\pi}\sigma} exp\left(\frac{-1}{2}\left(\frac{x}{\sigma}\right)^2\right)$dx.

$$\begin{aligned}
\text{Put } \tfrac{x}{\sigma} &= y \\
\Rightarrow \tfrac{1}{\sigma}\,\text{dx} &= \text{dy} \\
& \frac{\sigma^4}{\sqrt{2\pi}} \int_0^\infty 2.y^4 e^{\frac{-y^2}{2}}\ \text{dy} \\
\text{Put } \tfrac{y^2}{2} &= z \\
y\ \text{dy} &= \text{dz and} \\
y &= (2z)^{\frac{2}{2}} \\
&= \frac{\sigma^4.2}{\sqrt{2\pi}}.2^{3/2} \int_0^\infty z^{3/2} e^{-z}\ \text{dz} \\
&= \frac{4\cdot\sigma^4}{\sqrt{\pi}}.\Gamma\left(\frac{3}{2}+1\right) \\
&= \frac{4\cdot\sigma^4}{\sqrt{\pi}} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) \\
&= 3\sigma^4
\end{aligned}$$

Ans 3. (a) To find: $P(X < 200|X > 150)$

$$\begin{aligned}
P(X < 200|X > 150) &= \frac{P(150<X<200)}{P(X>150)} \\
&= \frac{\int_{150}^{200} \frac{100}{x^2}\ \text{dx}}{\int_{150}^{\infty} \frac{100}{x^3}\ \text{dx}} \\
&= \frac{\frac{1}{6}}{\frac{2}{3}} \\
&= \frac{1}{4}
\end{aligned}$$

(b) P(tube has life less than 150 hours)

$= P(X < 150) = 1 - \frac{2}{3} = \frac{1}{3}$

P(exactly one of 3 tubes is replaced after 150 hours)

$= {}^3C_1(\frac{1}{3})(\frac{2}{3})^2 = \frac{4}{9}$

(c) Let the number of tubes that may be inserted be

P(None of them is functioning after 150 hours)

$= \left(\frac{1}{3}\right)^n$

It is required that n be such that $\left(\frac{1}{3}\right)^n < 0.25$.

By hit and trial, $n = 2$.

Ans 4. (a) We know: $f_X(x) = F'_X(x) \Rightarrow f_X(x) = 4e^{-4x} : x > 0$

This is exponential Distribution with parameter 4.

(b) $\text{P}(X < 12\ minutes) = \text{P}\left(X < \frac{12}{60}\ hrs\right) = \text{P}\left(X < \frac{1}{5}\right) = 0.55067$

(c) $\text{P}(X < \frac{30}{60} | X > \frac{15}{60}) = \text{P}(X < \frac{1}{2} | X > \frac{1}{4}) = \frac{F_X(\frac{1}{2}) - F_X(\frac{1}{4})}{1 - F_X(\frac{1}{4})} = 0.63212$

Ans 5. $\text{P}(X > r + s | X > r) = \text{P}(X > s)$

$$\begin{aligned} &= \frac{P(X>r+s)}{P(X>r)} \\ &= \frac{e^{-\lambda(r+s)}}{e^{-\lambda r}} \qquad (\because F_X(x) = 1 - e^{-\lambda x}) \\ &= e^{-\lambda s} \\ &= P(X > s) \end{aligned}$$

Hence proved.