Self Evaluation Test

1. Obtain an orthonormal basis for v, the space of all real polynomials of degree atmost 2, the inner product being defined by,

$$(f,g) = \int_0^1 f(x)g(x)dx \qquad ...(1)$$

Solution. We have $v = \{a_0 + a_1x + a_2x^2 : a_i \in R\}$

Clearly $\{v_1 = 1,\ v_2 = x,\ v_3 = x^2\}$ is a basis of V.

Let $w_1 = v_1$ so that $||w_1||^2 = (w_1|w_1) = (1|1)$ or

$$||w_1||^2 = \int_0^1 1.1dx \qquad \text{Using(1)}$$

$$\therefore\ \frac{w_1}{||w_1||} = 1 \text{ let } w_2 = v_2 - \frac{(v_2|w_1)w_1}{||w_1||^2} \qquad (2)$$

We have, $(v_2, w_1) = (v_2, v_1) = \int_0^1 x.1dx = \frac{1}{2}$ So by (2) we get,

$$\begin{aligned} w_2 &= x - \frac{1}{2} \\ \Rightarrow ||w_2||^2 &= \int_0^1 w_2.w_2dx \\ &= \int_0^1 \left(x - \frac{1}{2}\right)^2 dx \\ &= \frac{1}{12} \\ \therefore\ \frac{w_2}{||w_2||} &= \sqrt{12}\left(x - \frac{1}{2}\right) \end{aligned}$$

$$\text{Let } w_3 = v_3 - \frac{(v_3|w_1)w}{||w_1||^2} - \frac{(v_3|w_2)w_2}{||w_2||^2} \qquad ...(3)$$

$$\begin{aligned} \text{We have, } (v_3|w_1) &= \int_0^1 v_3.w_1dx \\ &= \int_0^1 x^2 \cdot 1dx \\ &= \frac{1}{3}, \\ (v_3|w_2) &= \int_0^1 v_3.w_2dx \\ &= \int_0^1 x^2\left(x - \frac{1}{2}\right)dx \\ &= \frac{1}{4} - \frac{1}{6} \\ &= \frac{1}{12} \\ \text{and } ||w_1^2|| &= 1, \\ ||w_2||^2 &= \frac{1}{12} \end{aligned}$$

putting in (3), we get,

$$\begin{aligned} w_3 &= x^2 - \frac{1}{3} \cdot 1 - \left(x - \frac{1}{2}\right) dx \\ &= x^2 - x + \frac{1}{6} \\ ||w_3||^2 &= (w_3|w_3) \\ &= \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx \\ &= \tfrac{1}{180} \\ \therefore \quad \frac{w_3}{||w_3||} &= \sqrt{180}\left(x^2 - x + \frac{1}{6}\right) \end{aligned}$$

Hence an orthonormal basis for V is, $\left\{1, 2+\sqrt{3}\left(x - \frac{1}{2}\right), 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right)\right\}$

2. Show that in a complex inner product space v. If x is orthogonal to y then $||x+y||^2 = ||x||^2 + ||y||^2$.

However the converse need not be true. Justify.

Solution. We have, $(x|y) = 0 \Rightarrow (y|x) = (\overline{x,y}) = \overline{0} = 0$

$$\begin{aligned} \therefore \quad ||x+y||^2 &= (x+y, x+y) \\ &= (x|x) + (x|y) + (y|x) + (y|y) \\ &= ||x||^2 + ||y||^2 \end{aligned}$$

However the converse need not be true.

Consider, $V = \mathbb{C}^2$ with standard inner product.

Let $x = (0, i)$ $y = (0, 1) \in V$ then

$(x|y) = 0.0 + i.1 = i \neq 0 \Rightarrow$ x is not orthogonal to y.

Now $||x||^2 = 0.0 + i.\bar{i} = i(-i) = 1$

$||y||^2 = 0.1 + 1.1 = 1$ We have, $(0, (1+i))$, and so $||x+y||^2 = 0.0 + (1+i)\overline{(1+i)}$

or $||x+y||^2 = (1+i)(1-i) = 1 - i^2 = 2$

$\Rightarrow$ $||x+y||^2 = ||x||^2 + ||y||^2$, but x is not orthogonal to y.

3. A 2×2 real symmetric matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ is positive definite iff the diagonal entries a and d are positive and the determinant $|A| = ad - bc = ad - b^2$ is positive.

Solution. To prove $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ is positive definite iff a and d are positive and $|A| = ad - b^2$ is positive.

Let $u = [x, y]^T$, then

$$
\begin{aligned}
f(u) &= u^T A u \\
&= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= ax^2 + 2bxy + dy^2
\end{aligned}
$$

Suppose $f(u) > 0$ for every $u \neq 0$, then $f(1,0) = a > 0$ and $f(0,1) = d > 0$.

Also we have $f(b,-a) = a(ad - b^2) > 0$ since $a > 0$, we get $ad - b^2 > 0$. Conversely suppose $a > 0, b = 0, ad - b^2 > 0$. Completing the square give us,

$$f(u) = a(x^2 + \frac{2b}{a}xy + \frac{b^2}{a}y^2) + dy^2 - \frac{b^2}{a}y^2$$
$$= a\left(x + \frac{by}{a}\right)^2 + \frac{ad - b^2}{a}y^2$$

Accordingly $f(u) > 0$ for every $u \neq 0$

4. Let $(x|(1))$ be complex inner product space, and let $\theta : X \to X$ be any linear map such that $(\theta v|v) = 0 \; \forall v \in X$ then $\theta = 0$, the zero map.

Solution. For all $x, y \in X$ and all $\alpha \in \mathbb{C}$ we have,

$$
\begin{aligned}
0 &= (\theta(\alpha x + y)|\alpha x + y) \\
&= (\theta(\alpha x) + \theta y|\alpha x + y) \\
&= \underbrace{(\theta(\alpha x)|\alpha x)}_{0} + (\theta(\alpha x)|y) + (\theta(y)|\alpha x) + \underbrace{(\theta y|y)}_{0} \\
&= (\alpha(\theta x)|y) + (\theta y|\alpha x) \\
&= \overline{\alpha}(\theta(x)|y) + \alpha(\theta(y)|x) \qquad \text{...(1)}
\end{aligned}
$$

Put first $\alpha = 1$ and then $\alpha = i$ in the equation (1), we get

$$
\begin{aligned}
(\theta(x)|y) + (\theta(y)|x) &= 0 \qquad \text{...(2)} \\
-i(\theta(x)|y) + i(\theta(y)|x) &= 0 \; \forall \; x, y \in X \qquad \text{...(3)}
\end{aligned}
$$

Applying $i(2) + (3)$, we get

$2i(\theta y|x) = 0$, $2i \neq 0 \Rightarrow (\theta y|x) = 0 \; \forall \; x, y \in X$

$\Rightarrow \theta(y) = 0 \; \forall \; y \in X$

$(\because (x_0, y) = 0 \; \forall \; y \in X$ iff $x_0 = 0)$

$\Rightarrow \theta = 0$, the zero map.

Note. This is not the case when $(X, (1))$ is a real inner product space, for instance let $\theta : \mathbb{R}^2 \to \mathbb{R}^2$ rotate each $x \in \mathbb{R}^2$ by $90°$.

5. Let V be a complex inner product space and let $T \in L(V)$. Then T is self adjoint iff

$$(Tv|v) \in \mathbb{R} \ \ \forall v \in V \ [T \text{ is self adjoint if } T^* = T].$$

Proof. Let $v \in V$. Then,

$$\begin{aligned} (Tv|v) - \overline{(Tv|v)} &= (Tv|v) - (v|Tv) \\ &= (Tv|v) - (T^*v|v) \\ &= ((T - T^*)v|v) \end{aligned}$$

If $(Tv|v) \in R$ $\forall v \in V$ then the left hand side of above equation becomes 0. So,

$$\begin{aligned} ((T - T^*)v|v) &= 0 \ \forall \ v \in V \\ \Rightarrow \ (T - T^*)v &= 0 \ \forall \ v \in V \\ \Rightarrow \ T - T^* &= 0 \\ \Rightarrow \ T &= T^* \end{aligned}$$

and hence T is self adjoint.

Conversely, if T is self adjoint then the right hand side of above equation becomes 0.

So $(Tv|v) = \overline{(Tv|v)}$ for every $v \in V$, this implies that $(Tv|v) \in R$ for every $v \in V$ as desired.