Self Evaluation Test

1. Which of the following functions $\beta : R^2 \times R^2 \to R$ are bilinear forms?

(a) $\beta(x, y) = 1,\ x = (x_1, x_2),\ \ y = (y_1, y_2)$

(b) $\beta(x, y) = (x_1 + y_1)^2 - (x_1 - y_1)^2$

(c) $\beta(x, y) = 1,\ x = (x_1 y_2) - (x_2 y_1)$

(R^2 is a R-module, $R \sim$ set of real no's)

Solution (a) $\beta(x, z) = 1,\ \beta(y, z) = 1,\ \beta(x + y, z) = 1 \neq \beta(x, z) + \beta(y, z)$

$\beta(x, y) = 1$ not a bilinear form.

(b)

$$\begin{aligned}
\beta(x, y) &= \cancel{x_1^2} + \cancel{y_1^2} + 2x_1y_1 - \cancel{x_1^2} - \cancel{y_1^2} + 2x_1y_1 \\
\beta(x, y) &= 4x_1y_1 \\
\beta(x\lambda, \mu y) &= \beta\left(\lambda(x_1, x_2), \mu(y_1, y_2)\right) \\
&= \beta\left((\lambda x_1, \lambda x_2), (\mu y_1, \mu y_2)\right) \\
&= 4\lambda x_1 \mu y_1 \\
&= \lambda\mu 4 x_1 y_1 \\
&= \lambda\mu\beta(x, y). \\
\beta(x + \mu, y) &= \beta\left((x_1, x_2) + (\mu_1, \mu_2), (y_1, y_2)\right) \\
&= \beta\left((x_1 + \mu_1, x_2 + \mu_2), (y_1, y_2)\right) \\
&= 4(x_1 + \mu_1)y_1 \\
&= 4x_1y_1 + \mu_1 y_1 \\
&= \beta(x, y) + \beta(\mu, y)
\end{aligned}$$

$\Rightarrow$ β is a bilinear form.

(c)

$$\begin{aligned}
\beta(x, y) &= x_1y_2 - x_2y_1 \\
\beta(\lambda x, \mu y) &= \beta\left((\lambda x_1, \lambda x_2), (\mu y_1, \mu y_2)\right) \\
&= \lambda x_1 \mu y_2 - \lambda x_2 \mu y_1 \\
&= \lambda\mu x_1 y_2 - \lambda\mu x_2 y_1 \\
&= \lambda\mu(x_1y_2 - x_2y_1) \\
&= \lambda\mu\beta(x, y).
\end{aligned}$$

$$\begin{aligned}
\beta(x+\mu, y) &= \beta\left((x_1+\mu_1, x_2+\mu_2), (y_1, y_2)\right) \\
&= (x_1+\mu_1)y_2 - (x_2+\mu_2)y_1 \\
&= x_1y_2 + \mu_1y_2 - x_2y_1 - \mu_2y_1 \\
&= x_1y_2 - x_2y_1 + \mu_1y_2 - \mu_2y_1 \\
&= \beta(x,y) + \beta(\mu, y)
\end{aligned}$$

$\Rightarrow$ β is a bilinear form.

2. The following expressions define quadratic forms Q on R^2. Find the symmetric bilinear form β corresponding to each Q.

(a) ax_1^2

(b) $3x_1x_2 - x_2^2$

Solution.(a)

$$\begin{aligned}
Q(x) &= ax_1^2 \\
&= \beta(x,x) \text{ where } x = (x_1, x_2) \\
\beta(x,y) &= \frac{1}{4}\left[Q(x+y) - Q(x-y)\right] \\
Q(x+y) &= Q((x_1+y_1), (x_2+y_2)) \\
&= a(x_1+y_1)^2 \\
Q(x-y) &= a(x_1-y_1)^2 \\
\beta(x,y) &= \frac{1}{4}\left[a(x_1+y_1)^2 - a(x_1-y_1)^2\right] \\
&= \frac{4a}{4}x_1y_1 \\
&= ax_1y_1
\end{aligned}$$

(b)

$$\begin{aligned}
Q(x) &= 3x_1x_2 - x_2^2 \\
Q((x_1+y_1), (x_2+y_2)) &= Q(x+y) \\
&= 3(x_1+y_1)(x_2+y_2) - (x_2+y_2)^2 \\
Q(x-y) &= Q((x_1-y_1), (x_2-y_2)) \\
&= 3(x_1-y_1)(x_2-y_2) - (x_2-y_2)^2 \\
\beta(x,y) &= \frac{1}{4}\left[Q(x+y) - Q(x-y)\right] \\
&= \frac{1}{4}\left[3(x_1+y_1)(x_2+y_2) - (x_2+y_2)^2 - 3(x_1-y_1)(x_2-y_2) + (x_2-y_2)^2\right] \\
&= \frac{1}{4}\left[6x_1y_2 + 6y_1x_2 - 4x_2y_2\right] \\
&= \frac{1}{2}\left[3x_1y_2 + 3y_1x_2 - 2x_2y_2\right]
\end{aligned}$$

3. Let Q be a quadratic form associated with symmetric bilinear form β. Verify the polar identity

$$\beta(x,y) = \frac{1}{2}\left[Q(x+y) - Q(x) - Q(y)\right].$$

Solution.

$$\begin{aligned}
\frac{1}{2}\left[Q(x+y) - Q(x) - Q(y)\right] &= \frac{1}{2}\left[\beta(x+y, x+y) - \beta(x,x) - \beta(y,y)\right] \\
&= \frac{1}{2}\left[\beta(x,x) + \beta(x,y) + \beta(y,x) + \beta(y,y) - \beta(x,x) - \beta(y,y)\right] \\
&= \frac{1}{2} \times 2\beta(x,y) \quad [\beta(x,y) = \beta(y,x)] \\
&= \beta(x,y)
\end{aligned}$$

4. Every bilinear form on a vector space X over a field F can be uniquely expressed as the sum of a symmetric and skew-symmetric bilinear forms.

Solution. We know that every vector space over a field is also a module.

Let β be a bilinear form on a vector space X over F.

Let $g(x,y) = \frac{1}{2}\left[\beta(x,y) + \beta(y,x)\right]$

$g(x,y) = \frac{1}{2}\left[\beta(x,y) - \beta(y,x)\right] \quad \forall\ x, y \in X$

Therefore g and h are also bilinear form on X.

$$\begin{aligned}
g(y,x) &= \frac{1}{2}\left[\beta(y,x) + \beta(x,y)\right] \\
&= g(x,y).
\end{aligned} \quad \Rightarrow\ g \text{ is symmetric.}$$

$$\begin{aligned}
h(y,x) &= \frac{1}{2}\left[\beta(y,x) - \beta(x,y)\right] \\
&= -\frac{1}{2}\left[\beta(x,y) - \beta(y,x)\right] \\
&= -h(x,y).
\end{aligned}$$

$\Rightarrow\ h$ is skew-symmetric.

$$\begin{aligned}
\beta(x,y) &= g(x,y) + h(x,y) \\
\Rightarrow\ \beta &= g + h.
\end{aligned}$$

Now suppose that $\beta = \beta_1 + \beta_2$ where β_1 is the symmetric bilinear form and β_2 is skew symmetric bilinear form.

$$\begin{aligned}
\beta(x,y) &= (\beta_1 + \beta_2)(x,y) \\
&= \beta_1(x,y) + \beta_2(x,y) \qquad \text{...(1)} \\
\beta(y,x) &= (\beta_1(y,x) + \beta_2(y,x)) \\
&= \beta_1(x,y) - \beta_2(x,y)) \qquad \text{...(2)}
\end{aligned}$$

Adding (1) and (2), we get

$$\begin{aligned} 2\beta_1(x,y) &= \beta(x,y)+\beta(y,x) \\ \beta_1(x,y) &= \frac{1}{2}\left[\beta(x,y)+\beta(y,x)\right] \\ &= g(x,y) \\ \Rightarrow\ \beta_1 &= g \end{aligned}$$

Similarly it can be proved that $\beta_2 = h$.

5. Can a sesquilinear is a bilinear form.

Solution. Only zero form is both bilinear and sesquilinear form. Non zero sesquilinear form can not be a bilinear form.

Suppose β is a sesquilinear form and bilinear form.

$$\begin{aligned} \beta(x,\lambda y) &= \overline{\lambda}\beta(x,y) \\ \beta(x,\lambda y) &= \lambda\beta(x,y) \\ (\overline{\lambda}-\lambda)\beta(x,y) &= 0 \ \text{ (In general } \overline{\lambda}-\lambda \neq 0) \\ \Rightarrow\ \beta(x,y) &= 0 \ \ \forall\, x,y \in X \end{aligned}$$

$\Rightarrow$ β is zero form.