Self Evaluation Test

1. let A be a 2×2, non zero complex number st, $N^2 = 0$ then prove that N is similar over $\mathbb{C}$ to $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

Solution. Let $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ be a Linear operator st: $[T]_B = A$; $\quad B = \{v_1, v_2\}$ is basis of V

Now $0 = A^2 = A.A = [T]_B[T]_B = [T]_B^2 \quad \Rightarrow T = 0$

as $A \neq 0 \Rightarrow T \neq 0$

Let λ be an eigen value of $T \Rightarrow \quad \exists\ 0 \neq v \in V$ st: $T(v) = \lambda v$

$\Rightarrow 0 = T^2(v) = \lambda^2 v$ but $v \neq 0 \quad \Rightarrow \lambda = 0, 0$

$\Rightarrow 0 = \lambda$ is only eigen value of T.

Let $\omega_0 = \{x \in V : T(x) = 0\} = \ker T$ be the eigen space corresponding to $\lambda = 0$.

Since $0 \neq v \in \omega_0 \Rightarrow \omega_0 \neq \{0\}$

$\Rightarrow dim\omega_0 = 1or2$; if $dim\omega_0 = 2 \Rightarrow dim\omega_0 = dimV \Rightarrow \omega_0 = V$

$\Rightarrow KerT = V \Rightarrow T = 0$

$\Rightarrow dim\omega_0 = 1$; let $\omega_0 = \langle \omega_2 \rangle \Rightarrow \exists$ a subspace ω' of V st

$V = \omega' \oplus \omega_0, \Rightarrow dim\omega' = 1$; let $\omega' =< \omega_1 >$

Then $< \omega_1, \omega_2 >$ is basis of V

as $T(\omega_1), T(\omega_2) \in V = \omega' \oplus \omega_0$

So let $T(\omega_1) = \alpha_1\omega_1 + \alpha_2\omega_2$

$T(\omega_2) = 0\omega_1 + 0\omega_2 \quad (\because \omega_2 \in \omega_0)$

But $T^2 = 0$

$\Rightarrow 0 = T^2(\omega_1) = T(\alpha_1\omega_1 + \alpha_2\omega_2)$

$= \alpha_1(\alpha_1\omega_1 + \alpha_2\omega_2) + \alpha_2.0$

$= \alpha_1^2\omega_1 + \alpha_1\alpha_2\omega_2$

$\Rightarrow \alpha_1 = 0, \alpha_2 \neq 0$

(because if $\alpha_2 = 0 \Rightarrow T(\omega_1) = \alpha_1\omega_1 \Rightarrow \omega_1 \in \omega' \cap \omega_0 = \{0\} \Rightarrow \omega_1 = 0$)

$\Rightarrow T(\omega_1) = \alpha_2\omega_2$

Now $B' = \{\alpha_2^{-1}\omega_1, \omega_2\}$ is basis of V

(because $a\alpha_2^{-1}\omega_1 + b\omega_2 = 0$

$\Rightarrow a\alpha_2^{-1} = 0 = b$

$\Rightarrow a = 0 = b \Rightarrow$ L.I hence basis because $dimV = 2$)

$T(\alpha_2^{-1}\omega_1) = \alpha_2^{-1}T(\omega_1) = \alpha_2^{-1}(\alpha_2\omega_2) = \omega_2 = 0.\alpha_2^{-1}\omega_1 + 1.\omega_2$

$T(\omega_2) = 0.\alpha_2^{-1}\omega_1 + 0.\omega_2$

$\Rightarrow [T]_{B'} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ over $\mathbb{C}$

2. let P be a operator on R^2 such that $P(x, y) = (x, 0)$ what is the minimal polynomial for P?

Solution. we are given that $P(x, y) = (x, 0) \forall (x, y) \in R^2$.....(1)

Let $c \in R$ be an eigen value of p then there exist some $(x, y) \neq (0, 0) \in R$ such that

$P(x, y) = c(x, y)$

$\Rightarrow (x, 0) = (cx, cy)$

$\Rightarrow cx = x, cy = 0$

$\Rightarrow x(c - 1) = 0, cy = 0$

If $c = 0$ then $(0, 1)$ is an eigen vector of p since

$P(0, 1) = (0, 0) = c(0, 1)$

If $c = 1$ then $(1, 0)$ is an eigen vector of P since

$P(1, 0) = (1, 0) = c(1, 0)$

Hence $0, 1$ are the eigen values of P and characteristic polynomial for P is

$f(x) = (x - 0)(x - 1) = x(x - 1)$

If $P(x) = x \Rightarrow p(P) = P$ and $P(x, y) = (x, 0) \neq (0, 0)$ for $x \neq 0$

$\therefore p(P) \neq 0$

If $p(x) = x - 1 \Rightarrow p(P) = P - I$ and

$(P - I)(x, y) = P(x, y) - I(x, y) = (x, 0) - (x, y) = (0, -y) \neq (0, 0)$ for $y \neq 0$

$\Rightarrow p(P) \neq 0$

If $p(x) = x(x - 1) = x^2 - x \Rightarrow p(P) = P^2 - P$ and

$(p^2 - P)(x, y) = P(P(x, y)) - P(x, y)$

$= P(x, 0) - (x, 0) = (x, 0) - (x, 0) = (0, 0) \forall (x, y) \in \mathbb{R}^2$

$\Rightarrow p(P) = 0$

Hence minimal polynomial for P is $x(x - 1)$.

3. Let V be the vector space of $n \times n$ matrices over the field $\mathbb{F}$. Let A be a fixed $n \times n$ matrix. Let T be a Linear operator on V defined by

$$T(B) = AB \ \forall B \in V \qquad ...(1)$$

Show that the minimal polynomial for T is the minimal polynomial for A.

Solution. Let $p(x) = x^n + a_1x^{n-1} + ... + a_n \in \mathbb{F}$ be the minimal polynomial for T and

$q(x) = x^m + b_1x^{m-1} + ... + b_m \in \mathbb{F}[x]$ the minimal polynomial for A then,

$p(T) = 0$ and $q(A) = 0...(2)$

by $(1) T(I) = AI = A$

$T^2(I) = T(T(I)) = T(A) = A^2$

Similarly $T^3(I) = A^3, ..., T^n(I) = A^n$ using the results, we see that

$0 = p(T)I = (T^n + a_1T^{n-1} + ... + a_nI)I$

$= A^n + a_1A^{n-1} + ... + a_nI = p(A)$

$\Rightarrow p(A) = 0$

Now we show that $\dfrac{q(x)}{p(x)}$.

let c be a root of $p(x)$ we can write

$p(x) = (x - c)q(x) + r(x)$ where $r(x) = 0$ or $deg\ r(x) < deg\ q(x)$

we have $p(A) = (A - cI)q(A) + r(A)$

$\Rightarrow r(A) = 0 \quad (\because p(A) = q(A) = 0)$

If $r(x) \neq 0 \Rightarrow deg\ r(x) < deg\ q(x)$ and $r(A) = 0$ contradict the minimality of $q(x)$ so $r(x) = 0$

$\Rightarrow p(x) = (x - c)q(x) \Rightarrow \dfrac{q(x)}{p(x)}$

Finally we show that $\dfrac{p(x)}{q(x)}$

We have $O = q(A)B$

$= (A^m + b_1A^{m-1} + ... + b_mI)B$

$= [T^m(I) + b_1T^{m-1}(I) + ... + b_mI]B$

$= T^mB + b_1T^{m-1}B + ... + b_mI)B$

$= q(T) = 0$

Since $p(x)$ is the minimal polynomial for T and $q(T) = 0$

so $\dfrac{p(x)}{q(x)}$

$\Rightarrow p(x) = q(x)$ (ic) minimal polynomial for T is the minimal polynomial for A.

4. Let T be a Linear operator on $\mathbb{R}^3$ which is represented in standard ordered basis by the matrix

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}.$$

Prove that T is diagonalizable by exhibiting a basis for $\mathbb{R}^3$ each vector of which is characteristic vector of T.

Solution. Characteristic equation of T is $det(A - xI) = 0$

$$\begin{bmatrix} -9-x & 4 & 4 \\ -8 & 3-x & 4 \\ -16 & 8 & 7-x \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -1-x & 4 & 4 \\ -1-x & 3-x & 4 \\ -1-x & 8 & 7-x \end{bmatrix} = 0 \quad \text{by } c_1 + c_2 + c_3$$

$$\text{or } -(1+x) \begin{bmatrix} 1 & 4 & 4 \\ 1 & 3-x & 4 \\ 1 & 8 & 7-x \end{bmatrix} = 0$$

$$\text{or } -(1+x) \begin{bmatrix} 1 & 4 & 4 \\ 0 & -1-x & 0 \\ 0 & 4 & 3-x \end{bmatrix} = 0$$

$$\text{or } (1+x)(1+x)(3-x) = 0$$

Hence the characteristic values of T are $3, -1, -1$. The characteristic vector corresponding to $x = 3$ is given by

$$(A - 3I)X = 0$$

$$\Rightarrow \begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or}\quad \begin{bmatrix} -4 & 4 & 0 \\ 0 & -8 & 4 \\ 0 & -8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by $R_1 \to R_1 - R_2;\quad R_2 \to R_2 - 2R_1;\quad R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} -4 & 4 & 0 \\ 0 & -8 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by $R_3 = R_3 - R_2$

$\Rightarrow -x_1 + x_2 = 0, \quad -2X_2 + x_3 = 0$

These equations are satisfied by $x_1 = 1,\ x_2 = 1,\ x_3 = 2$. an eigen vector corresponding to eigen value $x = 3$ is

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

The eigen vector corresponding to the given value $x = -1$ is given by

$$(A + I)(X) = 0$$

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or}\quad \begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by $R_2 \to R_2 - R_1 \quad R_3 \to R_3 - 2R_1$

from the above equation we get

$-2x_1 + x_2 + x_3 = 0$ taking $x_2 = 0$, we get $x_1 = 1, x_3 = 2$

taking $x_3 = 0$ *we get* $x_1 = 1, x_2 = 2$

Hence , 2 L.I characteristic vectors corresponding to characteristic values $x = -1$ are $X_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad X_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

clearly; $X_1,\ X_2,\ X_3$ are linearly independent over $\mathbb{R}$ and so the set $\{X_1, X_2, X_3\}$

constitutes a basis of $\mathbb{R}^3$.

Hence T is diagonalizable. indeed for

$$p = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix};$$

$$P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

5. Find the characteristic polynomials for the identity operator and zero operator on an $n-$ dimensional vector space.

Solution. The characteristic polynomial of the identity operator I on V is

$$det(I - xI) = \begin{bmatrix} 1-x & 0 & - & - & - & 0 \\ 0 & 1-x & - & - & - & 0 \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ 0 & 0 & - & - & - & 1-x \end{bmatrix} = (1-x)^n$$

The characteristic polynomial of the zero operator O in V is

$$det(O - xI) = \begin{bmatrix} -x & 0 & - & - & - & 0 \\ 0 & -x & - & - & - & 0 \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ 0 & 0 & - & - & - & -x \end{bmatrix} = (-1)^n x^n.$$