Self Evaluation Test

1. Let λ be an eigen value of a linear operator T on a vector space $V(\mathbb{F})$. Let V_λ denote the set of all eigen vectors of T corresponding to eigen value λ. Prove that V_λ is a subspace of $V(\mathbb{F})$.

Solution. Here $V_\lambda = \{v \in V \mid v \text{ is an eigen vector of } T\}$.

$= \{v \in V | T(v) = \lambda v\}$.

Given is that λ be an eigen value of T.

$\therefore\ \exists$ a non zero vector v' such that $T(v') = \lambda v'$ so that $v' \in V_\lambda \Rightarrow V_\lambda \neq \phi$

i.e. V_λ is non-empty set.

let $v_1, v_2 \in V_\lambda$ and $\alpha, \beta \in \mathbb{F}$

Since $v_1, v_2 \in V_\lambda \Rightarrow Tv_1 = \lambda v_1$ and $Tv_2 = \lambda v_2$

$$\begin{aligned} \text{Now}\quad T(\alpha v_1 + \beta v_2) &= T(\alpha v_1) + T(\beta v_2) \\ &= \alpha T(v_1) + \beta T(v_2) \\ &= \alpha\lambda(v_1) + \beta\lambda(v_2) \\ &= \lambda(\alpha v_1 + \beta v_2) \end{aligned}$$

$\therefore\ T(\alpha v_1 + \beta v_2) = \lambda(\alpha v_1 + \beta v_2)$

$\Rightarrow\ \alpha v_1 + \beta v_2$ is an eigen vector corresponding to eigen value λ

$\Rightarrow\ \alpha v_1 + \beta v_2 \in V_\lambda$

Hence V_λ is a subspace of V.

2. Prove that the non zero eigen vectors corresponding to distinct eigen values of a linear operator are linearly independent.

Solution. let $v_1, v_2, \ldots, v_m$ be m non-zero eigen vectors of a linear operator $T : V \to V$ corresponding to distinct eigen values $\lambda_1, \lambda_2, \ldots, \lambda_m$ respectively.

$$\therefore\ T(v_1) = \lambda_1 v_1,\ T(v_2) = \lambda_2 v_2,\ \ldots,\ T(v_m) = \lambda_m v_m \qquad (1)$$

We want to show that $v_1, v_2, \ldots, v_m$ are L.I. vectors. We shall prove this result by induction on m.

Step I. Let $m = 1$

Then v_1 is L.I. since v_1 is a non-zero vector.

$\therefore$ the result is true for $m = 1$.

Step II. Assume the result is true for the number of vectors less than m.

Step III. Now, we shall show the result is true for m vectors.

$$\begin{aligned}
\text{Let} \quad a_1v_1 + a_2v_2 + \ldots + a_mv_m &= 0 \qquad (2)\\
\Rightarrow T(a_1v_1 + a_2v_2 + \ldots + a_mv_m) &= T(0)\\
\Rightarrow a_1T(v_1) + a_2T(v_2) + \ldots + a_mT(v_m) &= 0 \;\; [\text{Since } T \text{ is a L.T.}]\\
\Rightarrow a_1(\lambda_1v_1) + a_2(\lambda_2v_2) + \ldots + a_mT(\lambda_mv_m) &= 0 \;\; [\text{Using (1)}]\\
\Rightarrow a_1\lambda_1v_1 + a_2\lambda_2v_2 + \ldots + a_m\lambda_mv_m &= 0 \qquad (3)
\end{aligned}$$

Multiplying (2) on both sides by λ_m, we get

$$a_1\lambda_mv_1 + a_2\lambda_mv_2 + \ldots + a_m\lambda_mv_m = 0 \qquad (4)$$

$\therefore$ eq(3)-eq(4) gives

$a_1(\lambda_1 - \lambda_m)v_1 + a_2(\lambda_2 - \lambda_m)v_2 + \ldots + a_{m-1}(\lambda_{m-1} - \lambda_m)v_{m-1} = 0$

$\Rightarrow\ a_1(\lambda_1 - \lambda_m) = 0,\ a_2(\lambda_2 - \lambda_m) = 0, \ldots, a_{m-1}(\lambda_{m-1} - \lambda_m) = 0$

($\because\ v_1, v_2, \ldots, v_{m-1}$ are L.I. because of Step II)

$\Rightarrow a_1 = 0, a_2 = 0, \ldots, a_{m-1} = 0$

($\because\ \lambda_i - \lambda_m \neq 0$ for $1 \leq i \leq m-1$ as λ_i are distinct)

putting these in (2), we get

$$\begin{aligned}
a_mv_m &= 0\\
\Rightarrow \quad a_m &= 0 \quad [\because v_m \neq 0]
\end{aligned}$$

Thus we have $a_1 = a_2 = \ldots = a_m = 0$

$\therefore$ the vectors $v_1, v_2, \ldots, v_m$ are L.I.

Hence the result

3. Let λ be an eigen value of an invertible operator T on a vector space $V(\mathbb{F})$. Prove that λ^{-1} is an eigen value of T^{-1}

Solution. Given T be invertible operator.

$\Rightarrow\ T$ is non-singular.

$\Rightarrow\ \exists$ an eigen value $\lambda \neq 0$.

$\Rightarrow\ \lambda^{-1}$ exists.

Since λ is an eigen value of T, therefore there exists a non zero vector $v \in V$ such that.

$$T(v) = \lambda v$$

operating T^{-1} on both sides

$\Rightarrow \; T^{-1}(T(v)) = T^{-1}(\lambda v)$

$\Rightarrow \; v = \lambda T^{-1}(v)$

$\Rightarrow \; \frac{1}{\lambda} v = T^{-1}(v)$

or $T^{-1}(v) = \frac{1}{\lambda} v = \lambda^{-1}(v)$

$\Rightarrow \; \lambda^{-1}$ is an eigen value of T^{-1}

Hence the result.

4. Let V be vector space of all real valued continuous functions. Let T be a L.O. on V such that $T(f(x)) = \int_0^x f(t)\ \mathsf{dt}$. Show T has no eigen values.

Solution. If λ is an eigen value of T, then by definition of eigen value, $\exists$ a non zero $f(x) \in V$ such that

$$T(f(x)) = \lambda f(x)$$

$$\Rightarrow \int_0^x f(t)\ \mathsf{dt} = \lambda\ f(x) \tag{1}$$

Differentiating both sides we get $f(x) = \lambda\ f'(x)$

$\Rightarrow \; \dfrac{f'(x)}{f(x)} = \dfrac{1}{\lambda}$

Integrating, we get

$\log f(x) = \dfrac{x}{\lambda} + C$, C is constant of integration

$\Rightarrow \; f(x) = e^{\frac{x}{\lambda}+C} = e^C e^{\frac{x}{\lambda}} = ae^{\frac{x}{\lambda}}$ say

$\therefore f(0) = ae^0 = a$

so that $f(x) = f(0)e^{\frac{x}{\lambda}}$ (ii)

changing variable x by t we have

$f(t) = f(0)\ e^{\frac{t}{\lambda}}$

integrating both sides from 0 to x we get

$\int_0^x f(t)\ \mathsf{dt} = f(0) \int_0^x e^{\frac{t}{\lambda}}\ \mathsf{dt}$

$\lambda f(x) = f(0) \left[\dfrac{e^{\frac{t}{\lambda}}}{\frac{1}{\lambda}} \right]_0^x$ (using (i) for L.H.S)

$\lambda f(0) e^{\frac{x}{\lambda}} = f(0)\lambda(e^{\frac{x}{\lambda}} - 1)$ (using (ii) for L.H.S)

$e^{\frac{x}{\lambda}} = e^{\frac{x}{\lambda}} - 1$

$\Rightarrow\ 0 = -1$ which is wrong.

So that T has no eigen values.

5. Let $T : V \to V$ be a Linear operator on a finite dimensional vector space $V(\mathbb{F})$. Prove that the number of eigen values of T cannot exceed the dimension of vector space $V(\mathbb{F})$.

Solution. Given V is a finite dimensional vector space over $\mathbb{F}$.

Let us assume $\dim V = n$.

Now λ is an eigen value of T iff det $(\lambda I - T) = 0$

i.e., the eigen values of T are the roots of equation

$$\det(xI - T) = 0 \tag{1}$$

Since $\dim V = n$, so any matrix representation of T is of order $n\times$.

$\therefore$ the matrix representation of $xI - T$ is also of order $n \times n$.

$\Rightarrow$ The det $(xI - T)$ is a polynomial of degree n in x.

But the eigen values of T are roots of this polynomial [because of (i)]

$\therefore$ number of eigen values cannot exceed the degree n of the polynomial $\det(xI - T)$.

Hence the number of eigen values of T cannot exceed the $\dim V$.